

Towards rigorous numerical verification of hyperbolic equilibria

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Hyperbolicity ... a concept concerned with structural stability
(Under the assumption of hyperbolicity) \Rightarrow many results

- **How to verify the hyperbolicity** of invariant sets ?

Our goal : We show

- A criterion of the hyperbolicity of equilibria in infinite dimensional dynamical systems.

- 1 Preliminaries
 - Hyperbolic equilibrium
- 2 Hyperbolicity verification
 - Preceding results and Our requirements
 - Hyperbolicity verification theorem
- 3 Application
 - Rigorous numerical verification methods
 - An example

Hyperbolic equilibrium.

We consider an evolutionary equation

$$\dot{u} = -Au + f(u) \quad (1)$$

on a (separable) Hilbert space X .

- A : positive-definite self-adjoint (generally, sectorial)
- A^{-1} : compact
- $f : D(A^\alpha) \rightarrow X : C^1$.
- $\forall B \subset D(A^\alpha) : \text{bounded} \Rightarrow f(B) \subset X : \text{bounded}$.

Definition

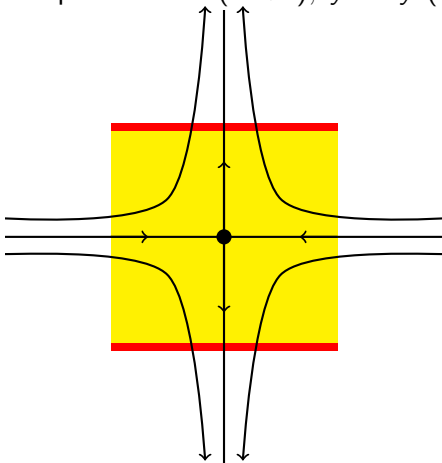
We say an equilibrium \bar{u} for (1) is *hyperbolic* if, for the linearization $L := -A + df(\bar{u})$ at \bar{u} ,

$$\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$$

holds, where $\sigma(L)$ is the spectrum of L .

Hyperbolic equilibrium

Example : $\dot{x} = ax$ ($a < 0$), $\dot{y} = by$ ($b > 0$)



$$(\sigma(L) =) \{a, b\} \cap \sqrt{-1}\mathbb{R} = \emptyset.$$

$\{0\}$ ($= S$) is hyp. equilibrium
with $\dim W^u(S) = 1$.

1 Preliminaries

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2 Hyperbolicity verification

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Preceding results for verifying equilibria of PDEs ...

- Existence
- Stability (e.g. Conley index)
- Uniqueness

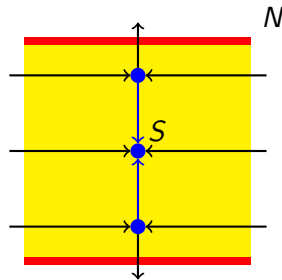
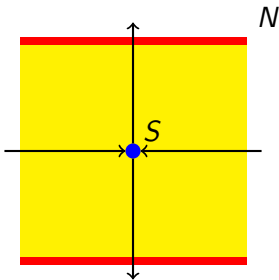
of equilibria **in a subset N** of a Banach space.

(Zgliczyński, Gameiro, Lessard, Mischaikow, ...)

(Nakao et. al., Oishi et. al.)

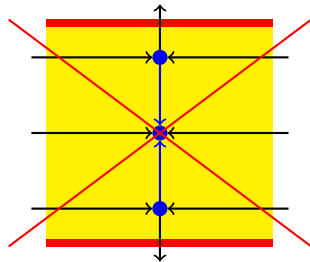
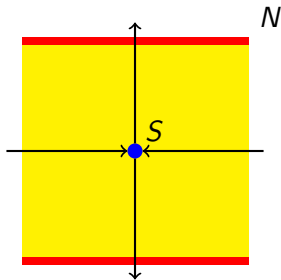
“Existence of equilibria”

(Schauder’s fixed point theorem, Conley index, Brouwer’s degree)



At least one equilibrium in N . Which is the true situation ?

“Existence + Local uniqueness”
(Contraction mapping principle)



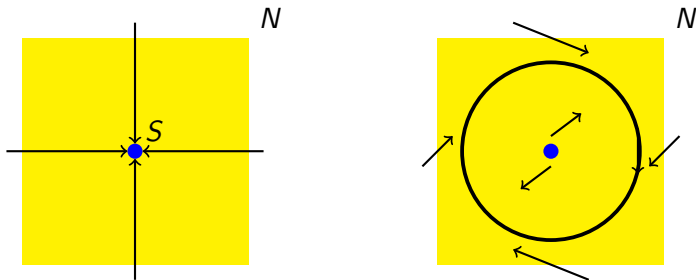
Equilibrium is unique in N . NOT the case of the right figure.

- **Hyperbolicity** : stability under perturbation of dynamical systems

As an invariant set of dynamical systems . . .

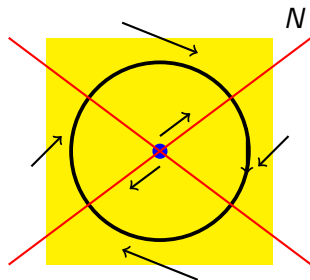
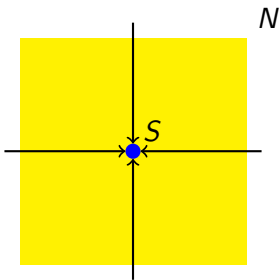
- Gradient dynamics : Inv. sets = equilibria + connecting orbits
- General dynamics : + “recurrent” inv. sets (e.g. periodic orbits, chaotic inv. sets)

Gradient dynamics vs. General dynamics



Even if an equilibrium \bar{u} is unique in N , $\{\bar{u}\} \neq \text{Inv}(N)$ in general.
 \Rightarrow **Stability of $\{\bar{u}\} \neq$ Stability of $\text{Inv}(N)$**

If dynamics in N is gradient-like ...



All recurrent invariant sets (e.g. periodic orbits) are excluded.
 \Rightarrow Precise structure of $Inv(N)$

Our requirements . . .

- Local uniqueness of equilibrium
- Hyperbolicity of equilibrium
- Construction of **Lyapunov function** on N
(so that dynamics is gradient-like in N)

Hyperbolicity verification theorem.

Two types of verification theorems
depending on rigorous numerical verification method.

- 1 Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) := d_i u_i + N_i(u), \quad i = 1, 2, \dots .$$

- 2 Other form.
(In case that u is not always expanded by eigenfunctions)

The Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi] \quad (2)$$

$$u(t, -x) = -u(t, x) \text{ with per. B.C.} \quad (3)$$

Using the Fourier basis $\{\sin(k\pi x)\}_{k \geq 1}$,

$u(t, x) = -2 \sum_{k \in \mathbb{N}} u_k \sin(k\pi x)$ and (2)+(3) is rewritten by

$$\dot{u}_k = k^2(1 - \nu k^2)u_k - k \sum_{n=1}^{k-1} u_n u_{k-n} + 2k \sum_{n=1}^{\infty} u_n u_{n+k},$$

which forms

$$\dot{u}_k = F_k(u) = d_k u_k + N_k(u), \quad k = 1, 2, \dots$$

Rigorous verification of equilibria

ZM-theory (Zgliczynski and Mischaikow, Found. Comp. Math. (2001) 255–288.)

Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \dots . \quad (4)$$

⇒ Find a set V (in a Hilbert space X) which forms

$$V = \prod_{k=1}^n [w_k^-, w_k^+] \times \prod_{k>n} \left[-\frac{C}{k^s}, \frac{C}{k^s} \right] \quad (w_k^\pm \in \mathbb{R}, C > 0, s \in \mathbb{N}) \quad (5)$$

containing an equilibrium u^* of (4).

(*Tool* : Conley-type index or mapping degree)

Hyperbolicity verification theorem.

We consider an evolutionary equation on X :

$$\dot{u} = F(u) \Leftrightarrow \dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \dots. \quad (6)$$

- $d_i < 0$ for all sufficiently large i .
- $u = (u_1, u_2, \dots) \in X$, $V \subset X$ which forms (5).
- $\partial F_i / \partial u_j \in C(V, \mathbb{R})$,
 $\sum_{j \geq 1} \max_{u \in V} |(\partial F_i / \partial u_j)(u)| \sup_{x, y \in V} |x_j - y_j| < \infty$.

Theorem (Zgliczyński)

If, for all $i \in \mathbb{N}$,

$$\inf_{u \in V} \left| \frac{\partial F_i}{\partial x_i}(u) \right| > \sum_{j \neq i} \sup_{u \in V} \left| \frac{\partial F_i}{\partial x_j}(u) \right|$$

holds, then $F(x) = 0$ has at most one solution in V .

Assumption

$$\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} = \emptyset \quad (7)$$

holds for all $u \in V$.

Theorem (M.)

Let u^* be an equilibrium of (6) in V . If (7) holds and

$$\inf_{i \in \mathbb{N}} [|d_i| - \sum_{j \geq 1} \sup_{u \in V} \left| \frac{\partial N_i}{\partial x_j}(u) \right|] = \delta_1 > 0, \quad (8)$$

$$\inf_{i \in \mathbb{N}} [|d_i| - \sum_{j \geq 1} \sup_{u \in V} \left| \frac{\partial N_j}{\partial x_i}(u) \right|] = \delta_2 > 0 \quad (9)$$

$$m := \text{the number of } d_i \text{ with positive real part} \quad (10)$$

hold, then $\text{Inv}(V) = \{u^*\}$. u^* is hyperbolic (for (6)) with $\dim W^u(\{u^*\}) = m$.

Outline of proof

Construct a Lyapunov function of the form

$$L(u) := - \sum_{i \geq 1} \text{sign}(d_i) \cdot (u_i - u_i^*)^2,$$

where u^ is the unique equilibrium of $\dot{u} = F(u)$ in V .*

- Define $G(u) := \frac{dL}{dt}(u)$.

Outline of proof

$$\frac{1}{2} \frac{\partial G}{\partial u_i} = -\text{sign}(d_i) \{ 2d_i(u_i - u_i^*) + (1 + \delta_{ij}) \sum_{j \geq 1} \frac{\partial N_i(c_{ij})}{\partial u_j} (u_j - u_j^*) \}$$

$$\frac{1}{2} \frac{\partial^2 G}{\partial u_i \partial u_j} = -\text{sign}(d_i) \left\{ 2d_i \delta_{ij} + \left(\frac{\partial N_i(c_{ij})}{\partial u_j} + \frac{\partial N_j(c_{ji})}{\partial u_i} \right) \right\}$$

By (8) and (9),

- u is the critical point of $G \Leftrightarrow u = u^*$.
- G is strictly negative-definite. $G(u) = 0$ iff $u = u^*$ (in V).
- $G \leq 0$. Thus L is a Lyapunov function in V .

In case of FEM...

Cannot expand u by *eigenfunctions*.

$$u = \sum_{i=1}^n u_i \varphi_i + \varphi_{\perp}, \quad \varphi_{\perp} \in S_h^{\perp}$$

- Behavior of Lyapunov function
⇒ perturbed diagonal system + *entrance condition* (M.)
- No information about spectrum in the tail term ... ?
⇒ *the Conley-type index* (M.)

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Rigorous numerical verification of hyperbolic equilibria

Existence of equilibria

ZM-theory

(or “FEM + the Conley-type index” (M.)).

Lyapunov function

Main theorem which is shown before.

Eigenvalue exclusion (= hyperbolicity)

Nakao's theory.

Rigorous numerical verification of hyperbolic equilibria

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$ for a linear operator $L \dots$

Assumption

L is *sectorial*. In particular, $\exists a \in \mathbb{R}, \varphi \in (0, \frac{\pi}{2})$ s.t.

$$S_{a,\varphi} := \{\lambda \in \mathbb{C} \mid \varphi \leq |\arg(\lambda - a)| \leq \pi\} \subset \rho(L).$$

$\Rightarrow S_{a,\varphi}^c \cap \sqrt{-1}\mathbb{R} = \text{"a finite interval"}$.

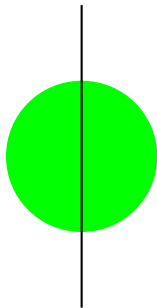
How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset \dots$

$$S_{a,\varphi}^c \cap \sqrt{-1}\mathbb{R}$$



How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset \dots$

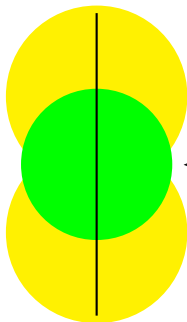
$$S_{a,\varphi}^c \cap \sqrt{-1}\mathbb{R}$$



⇐ 1st exclusion : no eigenvalues

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset \dots$

$$S_{a,\varphi}^c \cap \sqrt{-1}\mathbb{R}$$



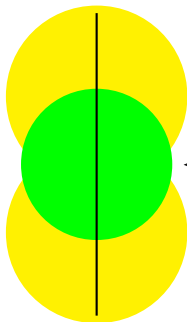
⇐ 2nd exclusion : no eigenvalues

⇐ 1st exclusion : no eigenvalues

⇐ 2nd exclusion : no eigenvalues

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset \dots$

$$S_{a,\varphi}^c \cap \sqrt{-1}\mathbb{R}$$



\Leftarrow 2nd exclusion : no eigenvalues

\Leftarrow 1st exclusion : no eigenvalues

\Leftarrow 2nd exclusion : no eigenvalues

Exclusion finished $\Rightarrow \sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$.

Ex. : the Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi],$$
$$u(t, x) = -u(t, -x), u(t, x + 2\pi) = u(t, x).$$

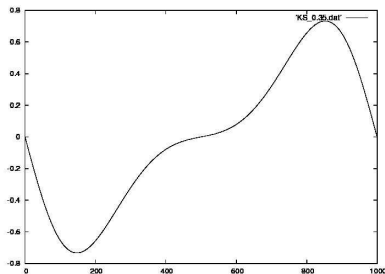
Remark

$$u^* \in \{u_{32}\} + \prod_{k=1}^{32} [-w_k, w_k] \times \prod_{k>32} \left[-\frac{C}{k^s}, \frac{C}{k^s} \right]$$

Equilibrium u_* of $u_t = -0.35u_{xxxx} - u_{xx} + 2uu_x$ in $[-\pi, \pi]$

$$C = 2040.74723, \quad s = 6, \quad \sup_{1 \leq k \leq 32} w_k \leq 3.59468318 \times 10^{-5}$$

$$\dim W^u(u^*) = 0.$$



Future works:

- Connecting orbits between stationary solutions of PDEs.

- Hyperbolicity + covering relation, cone condition
⇒ description of stable and unstable manifolds
- Known numerical verification methods
⇒ ODE-like methods to PDEs.

- Hyperbolic periodic orbits of PDEs.

- Rigorous computation of Poincaré maps (Zgliczyński)
- Topological tools (covering relation + discrete Conley index)
⇒ computation of $\dim W^u(\gamma)$ (M.)

⇒ GLOBAL DYNAMICS

⇒ Further applications (Bifurcation problems)