Towards rigorous numerical verification of hyperbolic equilibria

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Hyperbolicity ... a concept concerned with structural stability
(Under the assumption of hyperbolicity) $\Rightarrow$ many results

- How to verify the hyperbolicity of invariant sets ?

Our goal : We show

- A criterion of the hyperbolicity of equilibria in infinite dimensional dynamical systems.

Hyperbolic equilibrium.
We consider an evolutionary equation

$$\dot{u} = -Au + f(u) \tag{1}$$

on a (separable) Hilbert space X.

- A : positive-definite self-adjoint (generally, sectorial)
- A^{-1} : compact
- $f : D(A^\alpha) \to X$: C^1.
- $\forall B \subset D(A^\alpha)$: bounded $\Rightarrow f(B) \subset X$: bounded.

Definition

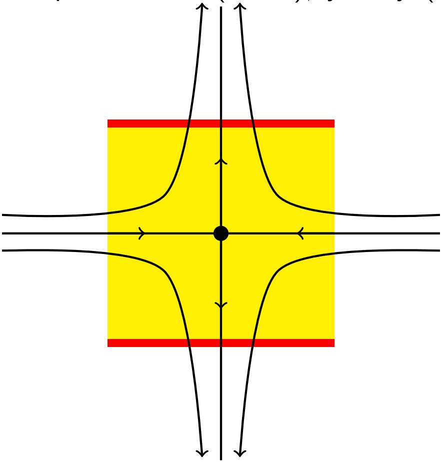
We say an equilibrium $\bar{u}$ for (1) is hyperbolic if, for the linearization $L := -A + df(\bar{u})$ at $\bar{u}$,

$$\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$$

holds, where $\sigma(L)$ is the spectrum of L.

Hyperbolic equilibrium

Example : $\dot{x} = ax\ (a < 0),\ \ \dot{y} = by\ (b > 0)$



$(\sigma(L) =)\{a, b\} \cap \sqrt{-1}\mathbb{R} = \emptyset.$

$\{0\}(= S)$ is hyp. equilibrium with $\dim W^u(S) = 1$.

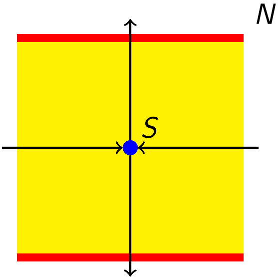
Preceding results for verifying equilibria of PDEs $\cdots$

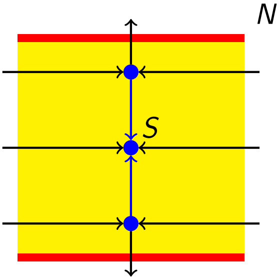
- Existence
- Stability (e.g. Conley index)
- Uniqueness

of equilibria in a subset N of a Banach space.

(Zgliczyński, Gameiro, Lessard, Mischaikow, ...)
(Nakao et. al., Oishi et. al.)

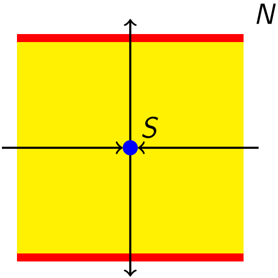
"Existence of equilibria"
(Schauder's fixed point theorem, Conley index, Brouwer's degree)

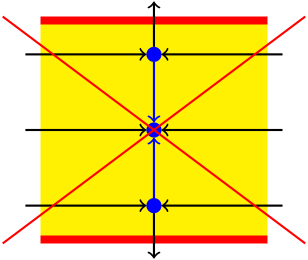




At least one equilibrium in N. Which is the true situation ?

"Existence + Local uniqueness"
(Contraction mapping principle)





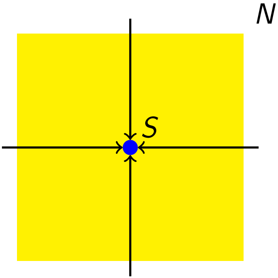
Equilibrium is unique in N. NOT the case of the right figure.

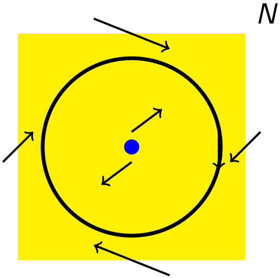
- Hyperbolicity : stability under perturbation of dynamical systems

As an invariant set of dynamical systems . . .

- Gradient dynamics : Inv. sets = equilibria + connecting orbits
- General dynamics : + "recurrent" inv. sets (e.g. periodic orbits, chaotic inv. sets)

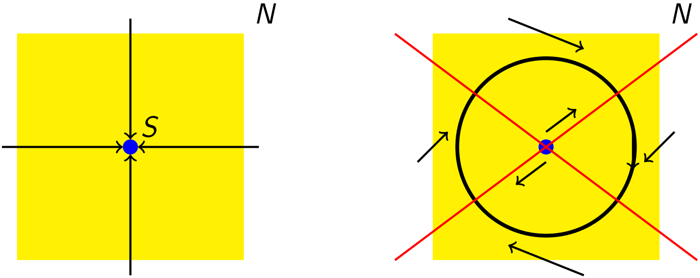
Gradient dynamics vs. General dynamics





Even if an equilibrium $\bar{u}$ is unique in N, $\{\bar{u}\} \neq Inv(N)$ in general.
$\Rightarrow$ Stability of $\{\bar{u}\} \neq$ Stability of $Inv(N)$

If dynamics in N is gradient-like . . .



All recurrent invariant sets (e.g. periodic orbits) are excluded.
$\Rightarrow$ Precise structure of $Inv(N)$

Our requirements . . .

- Local uniqueness of equilibrium
- Hyperbolicity of equilibrium
- Construction of Lyapunov function on N
  (so that dynamics is gradient-like in N)

Hyperbolicity verification theorem.

Two types of verification theorems
depending on rigorous numerical verification method.

1. Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) := d_i u_i + N_i(u), \quad i = 1, 2, \cdots .$$

2. Other form.
   (In case that u is not always expanded by eigenfunctions)

The Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi] \tag{2}$$

$$u(t,-x) = -u(t,x) \text{ with per. B.C.} \tag{3}$$

Using the Fourier basis $\{\sin(k\pi x)\}_{k\geq 1}$,
$u(t,x) = -2\sum_{k\in\mathbb{N}} u_k \sin(k\pi x)$ and (2)+(3) is rewritten by

$$\dot{u}_k = k^2(1-\nu k^2)u_k - k\sum_{n=1}^{k-1} u_n u_{k-n} + 2k\sum_{n=1}^{\infty} u_n u_{n+k},$$

which forms

$$\dot{u}_k = F_k(u) = d_k u_k + N_k(u), \quad k = 1, 2, \cdots .$$

Rigorous verification of equilibria

ZM-theory (Zgliczynski and Mischaikow, Found. Comp. Math. (2001) 255–288.)

Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \cdots . \tag{4}$$

$\Rightarrow$ Find a set V (in a Hilbert space X) which forms

$$V = \prod_{k=1}^{n} [w_k^-, w_k^+] \times \prod_{k>n} \left[-\frac{C}{k^s}, \frac{C}{k^s} \right] \quad (w_k^{\pm} \in \mathbb{R}, C > 0, s \in \mathbb{N}) \tag{5}$$

containing an equilibrium u^* of (4).
(*Tool* : Conley-type index or mapping degree)

Hyperbolicity verification theorem.

We consider an evolutionary equation on X:

$$\dot{u} = F(u) \Leftrightarrow \dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \cdots . \tag{6}$$

- $d_i < 0$ for all sufficiently large i.
- $u = (u_1, u_2, \cdots) \in X, \quad V \subset X$ which forms (5).
- $\partial F_i / \partial u_j \in C(V, \mathbb{R})$,
  $\sum_{j\geq 1} \max_{u\in V} |(\partial F_i/\partial u_j)(u)| \sup_{x,y\in V} |x_j - y_j| < \infty$.

Theorem (Zgliczyński)

If, for all $i \in \mathbb{N}$,

$$\inf_{u\in V} \left| \frac{\partial F_i}{\partial x_i}(u) \right| > \sum_{j\neq i} \sup_{u\in V} \left| \frac{\partial F_i}{\partial x_j}(u) \right|$$

holds, then $F(x) = 0$ has at most one solution in V.

Assumption

$$\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} = \emptyset \tag{7}$$

holds for all $u \in V$.

Theorem (M.)

Let u^* be an equilibrium of (6) in V. If (7) holds and

$$\inf_{i\in\mathbb{N}}[|d_i| - \sum_{j\geq 1} \sup_{u\in V} \left|\frac{\partial N_i}{\partial x_j}(u)\right|] = \delta_1 > 0, \tag{8}$$

$$\inf_{i\in\mathbb{N}}[|d_i| - \sum_{j\geq 1} \sup_{u\in V} \left|\frac{\partial N_j}{\partial x_i}(u)\right|] = \delta_2 > 0 \tag{9}$$

$$m := \text{ the number of } d_i \text{ with positive real part} \tag{10}$$

hold, then $Inv(V) = \{u^*\}$. u^* is hyperbolic (for (6)) with $\dim W^u(\{u^*\}) = m$.

Outline of proof

Construct a Lyapunov function of the form

$$L(u) := -\sum_{i\geq 1} sign(d_i)\cdot (u_i - u_i^*)^2,$$

where u^* is the unique equilibrium of $\dot{u} = F(u)$ in V.

- *Define $G(u) := \frac{dL}{dt}(u)$.*

Outline of proof

$$\frac{1}{2}\frac{\partial G}{\partial u_i} = -sign(d_i)\{2d_i(u_i - u_i^*) + (1+\delta_{ij})\sum_{j\geq 1}\frac{\partial N_i(c_{ij})}{\partial u_j}(u_j - u_j^*)\}$$

$$\frac{1}{2}\frac{\partial^2 G}{\partial u_i \partial u_j} = -sign(d_i)\left\{2d_i\delta_{ij} + \left(\frac{\partial N_i(c_{ij})}{\partial u_j} + \frac{\partial N_j(c_{ji})}{\partial u_i}\right)\right\}$$

By (8) and (9),

- *u is the critical point of $G \Leftrightarrow u = u^*$.*
- *G is strictly negative-definite. $G(u) = 0$ iff $u = u^*$ (in V).*
- *$G \leq 0$. Thus L is a Lyapunov function in V.*

In case of FEM...
Cannot expand u by *eigenfunctions*.

$$u = \sum_{i=1}^{n} u_i \varphi_i + \varphi_\perp, \quad \varphi_\perp \in S_h^\perp$$

- Behavior of Lyapunov function
  $\Rightarrow$ perturbed diagonal system + *entrance condition* (M.)
- No information about spectrum in the tail term ... ?
  $\Rightarrow$ *the Conley-type index* (M.)

Rigorous numerical verification of hyperbolic equilibria

Existence of equilibria

ZM-theory
(or “FEM + the Conley-type index” (M.)).

Lyapunov function

Main theorem which is shown before.

Eigenvalue exclusion (= hyperbolicity)

Nakao’s theory.

Rigorous numerical verification of hyperbolic equilibria

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$ for a linear operator $L \ldots$

Assumption

L is sectorial. In particular, $\exists a \in \mathbb{R}$, $\varphi \in (0, \frac{\pi}{2})$ s.t.

$$S_{a,\varphi} := \{\lambda \in \mathbb{C} \mid \varphi \leq |\arg(\lambda - a)| \leq \pi\} \subset \rho(L).$$

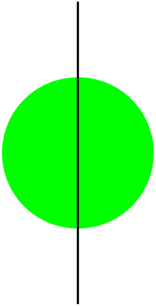
$\Rightarrow S^c_{a,\varphi} \cap \sqrt{-1}\mathbb{R} =$ "a finite interval".

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$ ...

$$S^c_{a,\varphi} \cap \sqrt{-1}\mathbb{R}$$

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$ ...

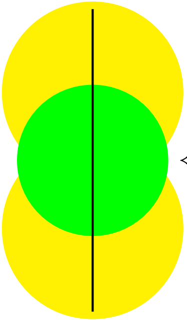
$S^c_{a,\varphi} \cap \sqrt{-1}\mathbb{R}$



$\Leftarrow$ 1st exclusion : no eigenvalues

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$ ...

$$S^c_{a,\varphi} \cap \sqrt{-1}\mathbb{R}$$



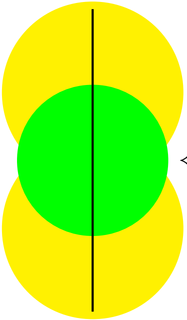
$\Leftarrow$ 2nd exclusion : no eigenvalues

$\Leftarrow$ 1st exclusion : no eigenvalues

$\Leftarrow$ 2nd exclusion : no eigenvalues

How to verify $\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$ ...

$S^c_{a,\varphi} \cap \sqrt{-1}\mathbb{R}$



$\Leftarrow$ 2nd exclusion : no eigenvalues

$\Leftarrow$ 1st exclusion : no eigenvalues

$\Leftarrow$ 2nd exclusion : no eigenvalues

Exclusion finished $\Rightarrow \sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$.

Ex. : the Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi],$$
$$u(t,x) = -u(t,-x), u(t, x+2\pi) = u(t,x).$$

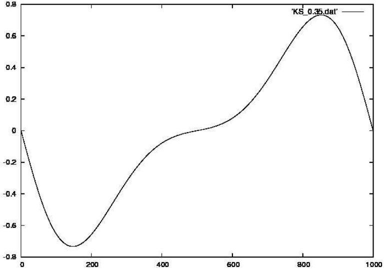
Remark

$$u^* \in \{u_{32}\} + \prod_{k=1}^{32} [-w_k, w_k] \times \prod_{k>32} \left[-\frac{C}{k^s}, \frac{C}{k^s} \right]$$

Equilibrium u_* of $u_t = -0.35u_{xxxx} - u_{xx} + 2uu_x$ in $[-\pi, \pi]$

$$C = 2040.74723, \;\; s = 6, \quad \sup_{1 \le k \le 32} w_k \le 3.59468318 \times 10^{-5}$$

$$\dim W^u(u^*) = 0.$$



Future works:

- Connecting orbits between stationary solutions of PDEs.
  - Hyperbolicity + covering relation, cone condition
    $\Rightarrow$ description of stable and unstable manifolds
  - Known numerical verification methods
    $\Rightarrow$ ODE-like methods to PDEs.
- Hyperbolic periodic orbits of PDEs.
  - Rigorous computation of Poincaré maps (Zgliczyński)
  - Topological tools (covering relation + discrete Conley index)
    $\Rightarrow$ computation of $\dim W^u(\gamma)$ (M.)

$\Rightarrow$ GLOBAL DYNAMICS

$\Rightarrow$ Further applications (Bifurcation problems)