Rigorous verification of bifurcations of differential equations via the Conley index theory

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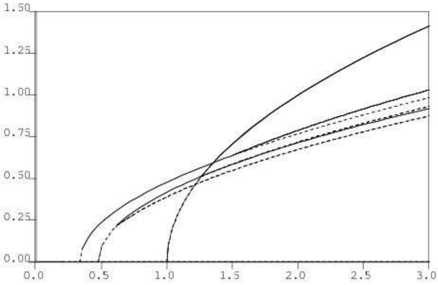
Our goal :

- We give a new *topological* method for rigorously capturing bifurcations.
- We apply the method to verifying bifurcations of concrete systems with rigorous computations.

Motivation : Bifurcation analysis of PDEs



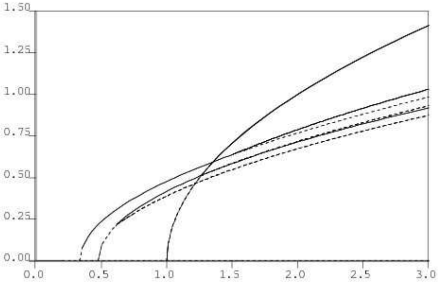
The Swift-Hohenberg equation

$$u_t = \left\{ \nu - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 \right\} u - u^3 \ ((t,x) \in [0,\infty) \times (-\pi/L, \pi/L)),$$

$$u(t, x + 2\pi/L) = u(t,x),\ u(t,-x) = u(t,x)$$

(Diagram : horizontal = ν-axis, vertical = $\|u\|_{L^2}$-axis)

Motivation : Bifurcation analysis of PDEs



A bifurcation occurs at $\nu \approx 0.62167$ ($L = 0.65$), which seems like a "pitchfork". Can we verify what occurs there *rigorously* ?

Another motivation : Analysis of discrete dynamical system with multiparameter variables
(Arai, Kalies, Kokubu, Mischaikow, Oka and Pilarczyk, *SIAM J. Appl. Dyn. Sys.* (2009), 757–789)

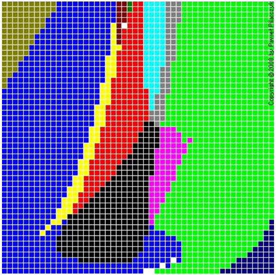
Nonlinear Leslie model :

$$T : (\mathbb{R} \geq 0)^2 \to (\mathbb{R} \geq 0)^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} (f_1 x_1 + f_2 x_2) e^{-0.1(x_1 + x_2)} \\ 0.7 x_1 \end{pmatrix},$$

$8 \leq f_1 \leq 37, 3 \leq f_2 \leq 50.$

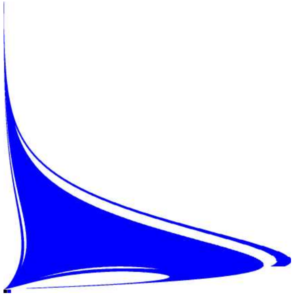
Another motivation : Analysis of discrete dynamical system with multiparameter variables

Continuation diagram for nonlinear Leslie model for
$(f_1, f_2) \in [8, 37] \times [3, 50]$.
(Divided into 50×50 grids)



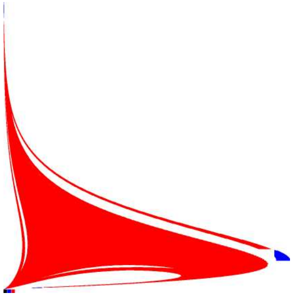
Another motivation : Analysis of discrete dynamical system with multiparameter variables

Outer approximation of invariant sets of nonlinear Leslie model at the box $(30, 2)$
(Phase : $(x_1, x_2) \in [-0.001, 320.056] \times [-0.001, 224.040]$)



Another motivation : Analysis of discrete dynamical system with multiparameter variables

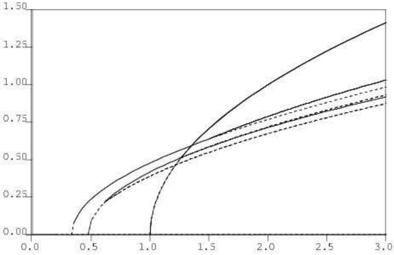
Outer approximation of invariant sets of nonlinear Leslie model at the box $(31, 2)$
(Phase : $(x_1, x_2) \in [-0.001, 320.056] \times [-0.001, 224.040]$)



We can consider that "a bifurcation" occurs at $(f1, f2)$ in the box $(30, 2)$ or $(31, 2)$.
(i.e. $f_1 \in [25.4, 26.56]$, $f_2 \in [4.88, 5.82]$)
$\Rightarrow$ How can we describe this phenomenon mathematically ?
$\Rightarrow$ A "topological approach" may be better than an analytic one.

Results :

- We give a *topological formulation* of pitchfork-type bifurcation.
- We verify a pitchfork-like bifurcation, in the sense of the topological formulation, of the SH eq. at $\nu \approx 0.62167$ by rigorous computations.

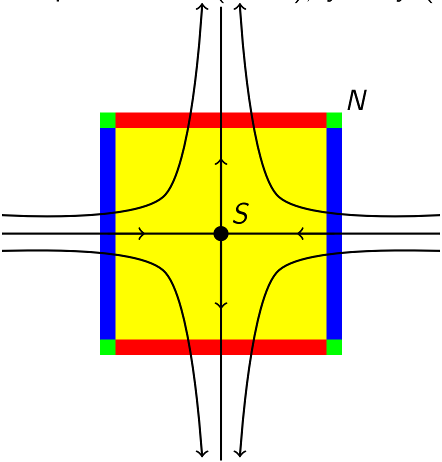


ν

$\nu \approx 0.62167$

An Isolating neighborhood.

Example : $\dot{x} = ax\ (a < 0),\ \ \dot{y} = by\ (b > 0)$



N : isolating neighborhood

S : isolated invarinat set

vertical lines : entrance

$N^+ :=$ the exit

$\quad = $ *horizontal lines* $+$ *vertices*

(N^+ is closed.)

$\Rightarrow N$ is an isolating block.

The pair (N, N^+) is called an “index pair”.

The Conley index :

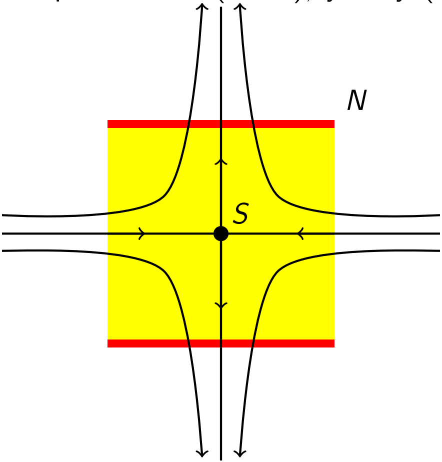
- φ : semiflow on a locally compact metric space X
- $S, N \subset X$: compact
- $Inv_\varphi(N) := \{x \in X \mid \exists \sigma : \mathbb{R} \to X$: solution s.t. $\sigma(0) = x,\ \sigma(\mathbb{R}) \subset N\}$
- N : isolating neighborhood $\Leftrightarrow Inv_\varphi(N) \subset int(N)$
- S : isolated invariant set $\Leftrightarrow \exists N$ isol. nbh. s.t. $S = Inv_\varphi(N)$.
  $\Rightarrow$ *The Conley index* of S is the homology group

$$CH_*(S, \varphi) = H_*(N_1/N_2, [N_2]).$$

$((N_1, N_2)$: an index pair of S in $N)$

The Conley index

Example : $\dot{x} = ax \ (a < 0), \ \dot{y} = by \ (b > 0)$



(N, N^+) : index pair

$$CH_n(S) \cong \begin{cases} R & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$(\approx$ Morse index$)$

Properties of the Conley index

- $CH_*(S, \varphi)$: independent of index pairs of S.
- $CH_*(S, \varphi) \not\cong 0 \Rightarrow S \neq \emptyset$.

$\Lambda := [\lambda_-, \lambda_+] \subset \mathbb{R}$
$\{\varphi^\lambda\}_{\lambda \in \Lambda}$: a family of semiflows on X (cont. on λ)

- N : an isol. nbh. for φ^{λ_0}, $\lambda_0 \in \Lambda$
  $\Rightarrow N$: an isol. nbh. for φ^λ, $\forall \lambda$ near λ_0.
- $CH_*(Inv(N), \varphi^\lambda) \equiv$ const., $\forall \lambda$ near λ_0.

Definition

- $Y \subset X$.

An ω-limit set of Y

$$\omega(Y,\varphi) := \bigcap_{s\geq 0} \overline{Y\cdot [s,\infty)}.$$

An α-limit set of Y

$$\alpha(Y,\varphi) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \bigcup_{y\in Y} H(t,y)},$$

$$H(t,x) = \{y\in X \mid \exists \sigma : (-\infty,0] \to X : \text{ a solution s.t. } \sigma(0)=x, \sigma(-t)=y\}.$$

Morse decomposition : A gradient-like decomposition of invariant sets.

Definition

- *φ : a semiflow on X*
- *S : An isol. inv. set for φ*

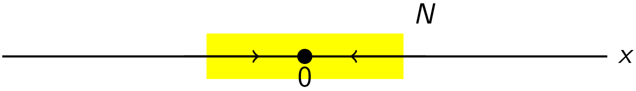
A collection of isol. inv. subsets $\{M(p) \mid p \in P : \text{finite}\}$ of S is a Morse decomposition of S if $\exists >$: strict partial order on P s.t. $\forall x \in S \setminus \bigcup_{p \in P} M(p)$, $\exists p, q \in P$ $(p < q)$ s.t.

$$\omega(x, \varphi) \subset M(p), \quad \alpha(x, \varphi) \subset M(q).$$

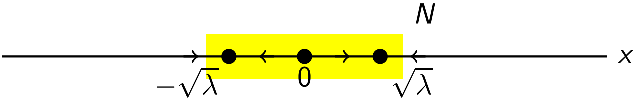
$>$: an *admissible ordering* on P.

Pitchfork bifurcation

$\dot{x} = \lambda x - x^3, \ \lambda < 0$



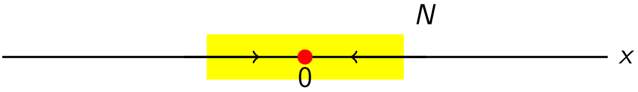
$\dot{x} = \lambda x - x^3, \ \lambda > 0$



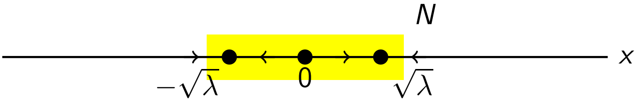
Example : $\dot{x} = \lambda x - x^3$. N : an isol. nbh.

Pitchfork bifurcation

$\dot{x} = \lambda x - x^3, \ \lambda < 0$



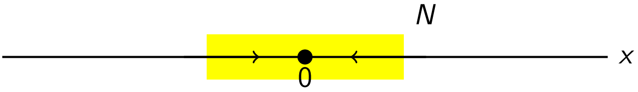
$\dot{x} = \lambda x - x^3, \ \lambda > 0$



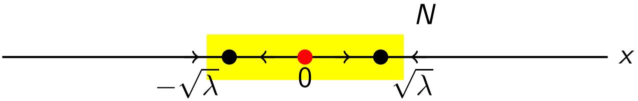
$\lambda < 0 : \ x = 0 :$ the only equilibrium (Morse index $= 0$)

Pitchfork bifurcation

$\dot{x} = \lambda x - x^3, \ \lambda < 0$



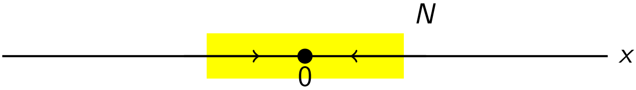
$\dot{x} = \lambda x - x^3, \ \lambda > 0$



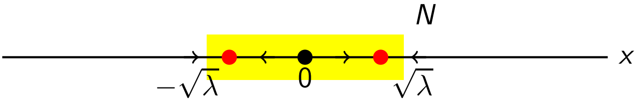
$\lambda > 0$: $x = 0$: an equilibrium (Morse index $= 1$)

Pitchfork bifurcation

$\dot{x} = \lambda x - x^3,\ \lambda < 0$



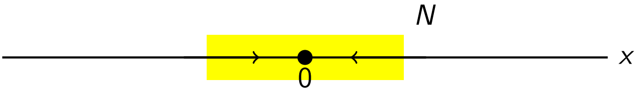
$\dot{x} = \lambda x - x^3,\ \lambda > 0$



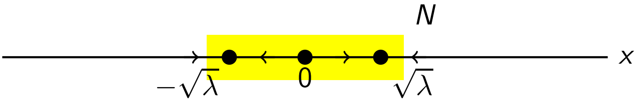
$\lambda > 0$: $x = \pm\sqrt{\lambda}$: equilibria (Morse index $= 0$)
$M(0^{\pm}) := \{\pm\sqrt{\lambda}\}$, $M(1) := \{0\}$. $\Rightarrow$ $\{M(0^-), M(0^+), M(1)\}$: a Morse dec. of $Inv(N)$ (adm. order. : $0^- < 1, 0^+ < 1$).

Pitchfork bifurcation

$\dot{x} = \lambda x - x^3, \ \lambda < 0$



$\dot{x} = \lambda x - x^3, \ \lambda > 0$



The equilibrium $x = 0$ BIFURCATES at $\lambda = 0$!
($\mathbb{Z}_2$-symmetry : $x \mapsto -x$)
N : isol. nbh., $\forall \lambda$ near $\lambda = 0$.

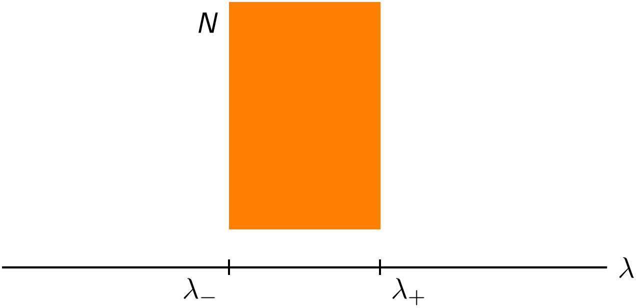
A C-pitchfork type isolating neighborhood :

- X : a locally compact metric space
- $\Lambda := [\lambda_-, \lambda_+] \subset \mathbb{R}$
- $\{\varphi^\lambda\}_{\lambda\in\Lambda}$: a family of semiflows on X (cont. on λ)
- $\mathbb{Z}_2$ acts on X.
- $\varphi^\lambda(t, gx) = g\varphi^\lambda(t, x)$, $g \in \mathbb{Z}_2$.
- $\Phi : [0, \infty) \times X \times \Lambda \to X \times \Lambda \quad (\Phi(t, x, \lambda) := (\varphi^\lambda(t, x), \lambda))$
- $N \subset X \times \Lambda \Rightarrow N^\lambda := N \cap (X \times \{\lambda\})$.

Definition

- *N : conn. isol. nbh. for Φ s.t. N^λ is connected $(\forall \lambda \in \Lambda)$ and that $\varphi^\lambda|_{Inv(N^\lambda)}$ is a flow.*

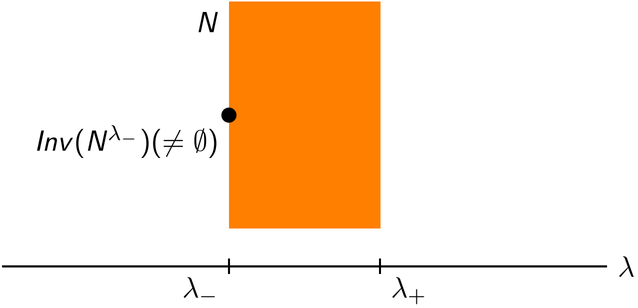
$\Rightarrow$ N is of C-pitchfork type over Λ if conditions (CPF1) - (CPF5) are satisfied.



Definition

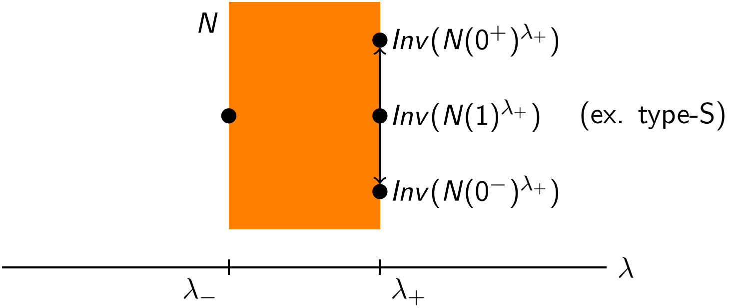
(CPF1)

- $CH_*(Inv(N^{\lambda_-}), \varphi^{\lambda_-}) \neq 0$.
- *NO Morse dec. of* $Inv(N^{\lambda_-})$ *containing* $\mathbb{Z}_2$*-asymmetric Morse comp.*



Definition

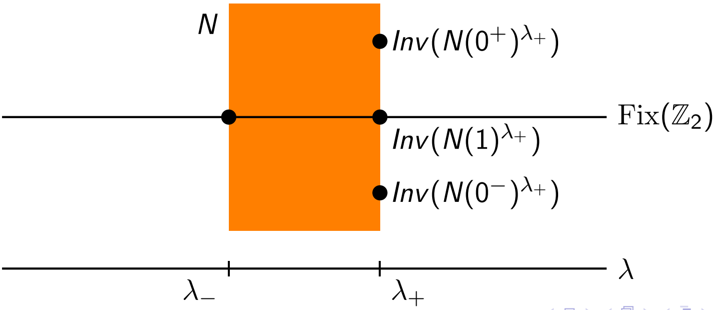
$\exists N(0^+)^{\lambda_+}$, $N(0^-)^{\lambda_+}$, $N(1)^{\lambda_+}$: mutually disjoint isol. nbhs. s.t.
(CPF2) $\{Inv(N(0^+)^{\lambda_+}), Inv(N(0^-)^{\lambda_+}), Inv(N(1)^{\lambda_+})\}$ *: a Morse dec. of $Inv(N^{\lambda_+})$ with one of the following adm. order.:*
(S) $0^+ < 1,\ 0^- < 1$ *or* *(U)* $0^+ > 1,\ 0^- > 1$.



Definition

(CPF3) *Inv*$(\tilde{N}(i)) := Inv(N(i)^{\lambda_+})$ $(i = 0^{\pm}, 1)$ *satisfy*

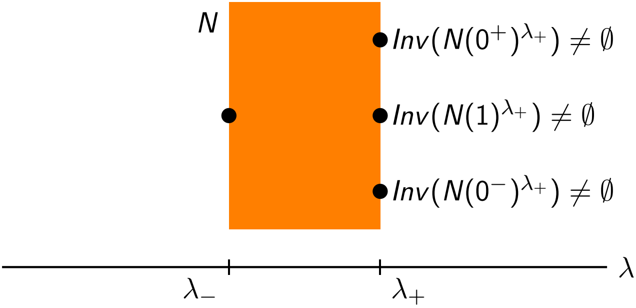
- $Inv(\tilde{N}(0^{\pm})) \cap \mathrm{Fix}(\mathbb{Z}_2) = \emptyset$.
- $gInv(\tilde{N}(0^+)) = Inv(\tilde{N}(0^-))$, $gInv(\tilde{N}(0^-)) = Inv(\tilde{N}(0^+))$,
- $gInv(\tilde{N}(1)) = Inv(\tilde{N}(1))$, $g \in \mathbb{Z}_2 \setminus \{id.\}$.



Definition

(CPF4) $CH_*(Inv(N(0^{\pm})^{\lambda_+}), \varphi^{\lambda_+}) \neq 0$.

(CPF5) $\exists S \subset Inv(N(1)^{\lambda_+})$: *isol. inv. set s.t.* $CH_*(S, \varphi^{\lambda_+}) \neq 0$.



The C-pitchfork bifurcation theorem

- $\Lambda_{PF} := \{\lambda \in \Lambda \mid \exists N(i)^\lambda\ (i = 0^\pm, 1)$: mutually disjoint isol. nbhs. s.t. $\{Inv(N(0^+)^\lambda), Inv(N(0^-)^\lambda), Inv(N(1)^\lambda)\}$ satisfies conditions (CPF2-5).$\}$
- $\lambda^* := \inf \Lambda_{PF}$.
- $\mathscr{M}_\lambda$: A Morse dec. of $Inv(N^\lambda)$ satisfying (CPF2-3).

Definition

- *S : a compact inv. set in X.*
- *$x, y \in S$.*

$\Rightarrow$ *An ϵ-chain from y to x (in S) is a sequence*

$$\{y = x_1, x_2, \cdots, x_{n+1} = x;\ t_1, t_2, \cdots, t_n \mid t_i \geq 1 \text{ for all } i\}$$

satisfying $d(x_i \cdot t_i, x_{i+1}) < \epsilon$ for all $i = 1, 2, \cdots, n$.

Theorem

N : isol. nbh. for Φ of C-pitchfork type over Λ.

(1) $\lambda^* < \lambda_+$.

(2) $\forall \lambda \in [\lambda_-, \lambda^*]$, $\forall \mathscr{M}_\lambda$ *satisfying (CPF5), we have*

$$CH_*(M(0^\pm)^\lambda, \varphi^\lambda) = 0.$$

If $\{M(p)\}_{p\in P}$ is an Morse dec. of $M(0^+)^\lambda$ or $M(0^-)^\lambda$ ($\lambda \leq \lambda^*$), then

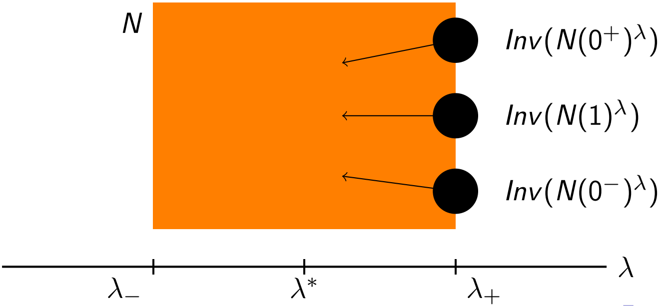
$$CH_*(M(p), \varphi^\lambda) = 0, \quad \forall p \in P.$$

Theorem

(3) *We define* $M(i)^{\lambda^*} := \bigcap_{\epsilon>0} \overline{\bigcup_{\lambda\in(\lambda^*,\lambda^*+\epsilon)} Inv(N(i)^\lambda)}$ $(i = 0^\pm, 1)$,

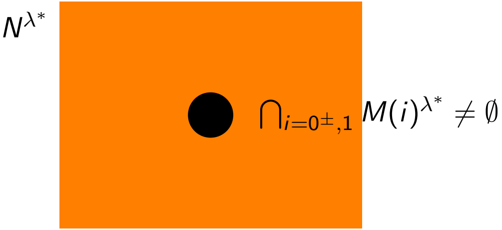
where $Inv(N(i)^\lambda)$ $(i = 0^\pm, 1)$ *are isol. inv. sets satisfying (CPF2-5).*

$\Rightarrow$ *One of the following statements holds.*



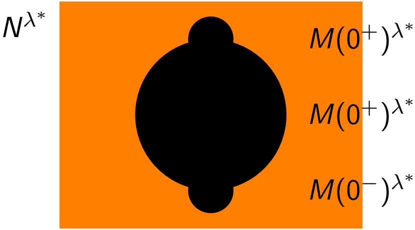
Theorem

Case 1. $\bigcap_{i=0^\pm,1} M(i)^{\lambda^*} \neq \emptyset$.



Theorem

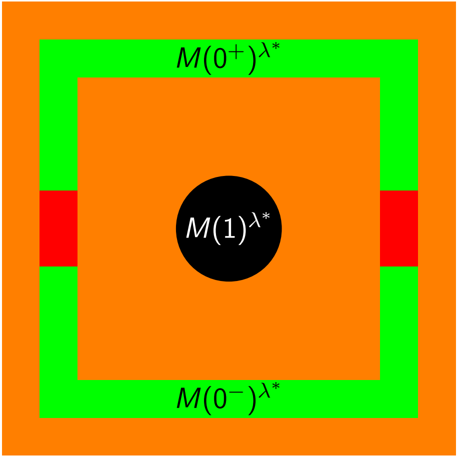
Case 2. $M(0^-)^{\lambda^*} \cap M(0^+)^{\lambda^*} = \emptyset$, $M(0^\pm)^{\lambda^*} \cap M(1)^{\lambda^*} \neq \emptyset$.



Theorem

Case 3. $M(0^-)^{\lambda^*} \cap M(0^+)^{\lambda^*} \neq \emptyset,\ M(0^\pm)^{\lambda^*} \cap M(1)^{\lambda^*} = \emptyset.$

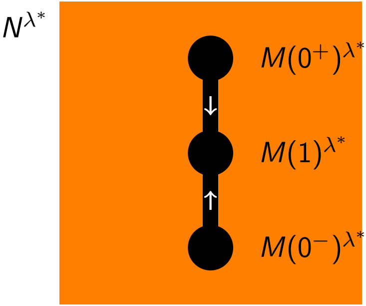
N^{λ^*}



Theorem

Case 4. *If $\lambda > \lambda^*$ is sufficiently close to λ^* and $M(i)^{\lambda^*}$ are mutually disjoint, then at least one of the following statements hold. For any $\epsilon > 0$,*

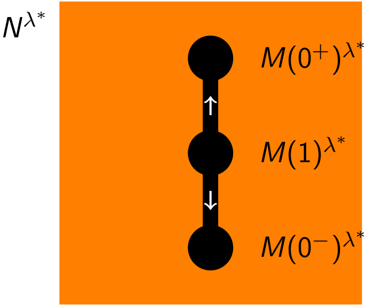
4-1. *$\exists$ an ϵ-chain from $M(0^\pm)^{\lambda^*}$ to $M(1)^{\lambda^*}$ (in case that $\mathscr{M}_\lambda$ has type-S ordering).*



Theorem

Case 4. If $\lambda > \lambda^*$ is sufficiently close to λ^* and $M(i)^{\lambda^*}$ are mutually disjoint, then at least one of the following statements hold. For any $\epsilon > 0$,

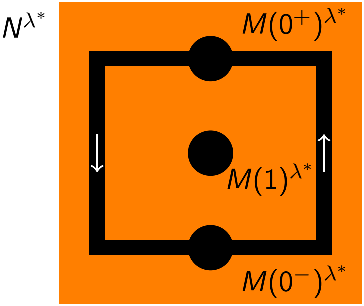
4-2. *$\exists$ an ϵ-chain from $M(1)^{\lambda^*}$ to $M(0^{\pm})^{\lambda^*}$ (in case that $\mathscr{M}_\lambda$ has type-U ordering).*



Theorem

Case 4. If $\lambda > \lambda^*$ is sufficiently close to λ^* and $M(i)^{\lambda^*}$ are mutually disjoint, then at least one of the following statements hold. For any $\epsilon > 0$,

4-3. *$\exists$ an ϵ-chain from $M(0^-)^{\lambda^*}$ to $M(0^+)^{\lambda^*}$ and from $M(0^+)^{\lambda^*}$ to $M(0^-)^{\lambda^*}$.*



- "Bifurcation" $\Rightarrow$ Change of recurrent and gradient structures of dynamics.
- "Existence of equilibria" $\Rightarrow$ Existence of isolated invariant sets with nontrivial Conley indices.
- "Pitchfork" $\Rightarrow$ Whether or not there are invariant sets with with nontrivial Conley indices which are $\mathbb{Z}_2$-asymmetric.
  - $\lambda \leq \lambda^*$: No.
  - $\lambda > \lambda^*$: Possible.

$\Rightarrow$ *C-pitchfork bifurcation*.

Outline of proof

(1). *It follows from the robustness of Morse dec.*

(2). *Assume* $\{M(p)\}_{p\in P}$ *: a Morse dec. of* $M(0^+)^\lambda$ *and* $\exists p_0 \in P$ *s.t.* $CH_*(M(p_0), \varphi^\lambda) \neq 0$. $\Rightarrow$ $\exists$ *a Morse dec. of* $Inv(N^\lambda, \varphi^\lambda)$ *satisfying (CPF2-5).* $\Rightarrow$ *Contradiction.*

(3). *If not, we can constract a Morse dec. of* $Inv(N^\lambda, \varphi^{\lambda^*})$ *satisfying (CPF2-5) by using* ϵ*-chains.* $\Rightarrow$ *Contradiction.*

The Swift-Hohenberg equation:

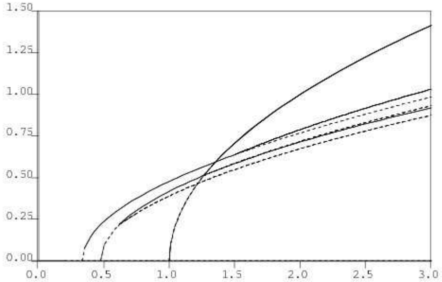
$$u_t = \left\{ \nu - \left(1 + \frac{\partial^2}{\partial x^2} \right)^2 \right\} u - u^3, \qquad (1)$$

$$u(t,-x) = u(t,x),\ \ u(t, x + 2\pi/L) = u(t,x),\ \ I = [-\pi/L, \pi/L].$$

- $\varphi_k(x) := \cos(kLx)$
- $u \in L^2(I)$: solution of (1) $\Rightarrow u(t) = \sum_{k=0}^{\infty} u_k(t)\varphi_k$.
- (1) is "equivalent" to

$$\dot{u}_k = (\nu - (1 - k^2L^2)^2)u_k - \sum_{n_1+n_2+n_3=k} u_{n_1}u_{n_2}u_{n_3},\ \ k = 0, 1, 2, \cdots .$$

Bifurcation from an nontrivial equilibrium

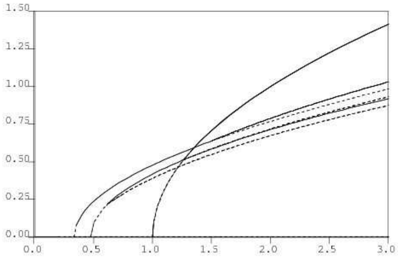


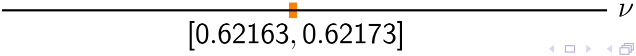
Bifurcation diagram of (1) for $L = 0.65$.
It is considered that a bifurcation occurs at $\nu \approx 0.62167$.
Let $\nu_- := 0.62163$ and $\nu_+ := 0.62173$.

Result : Bifurcation from an nontrivial equilibrium

Computer assisted result

The equation (1) admits a type-U C-pitchfork isolating neighborhood N_U over $(\nu \in)[0.62163, 0.62173]$, *which does not contain the trivial equilibrium.*





Conclusion :

- We have given a new notion for capturing bifurcations in terms of the Conley index and Morse decompositions.
    - C-pitchfork bifurcation
    - C-saddle-node bifurcation (a topological formulation of the saddle-node bifurcation)
- We have applied the notion to verifying bifurcations of a parabolic PDE.

Future works :

- Other bifurcations like
    - Hopf bifurcation
    - Bifurcations of more general invariant sets $\Rightarrow$ Effective algorithm for constructing C-type isolating neighborhoods
- Rigorous verification method for various problems (e.g. PDEs with various boundary conditions on arbitrary bounded domains)

Proof of the C-PF bifurcation theorem

Outline of proof

(1). *It follows from the robustness of Morse decompositions.*

(2). *If* $\{M(p)\}_{p \in P}$ *is a Morse dec. of* $M(0^+)^\lambda$ *and there exists* $p_0 \in P$ *such that* $CH_*(M(p_0), \varphi^\lambda) \neq 0$. *Then we can reconstruct a Morse decomposition of* $Inv(N^\lambda, \varphi^\lambda)$ *satisfying (CPF2-5), which can be proved by "Conley's homology exact sequences". This contradicts the definition of* λ^*.

Proof of the C-PF bifurcation theorem

Outline of proof

(3) *We prove that the last case [4] holds if $M(i)^{\lambda^*}$ are mutually disjoint, where*

$$M(i)^{\lambda^*} := \bigcap_{\epsilon>0} \overline{\bigcup_{\lambda \in (\lambda^*, \lambda^*+\epsilon)} Inv(N(i)^{\lambda})} \ (i = 0^{\pm}, 1),$$

and $Inv(N(i)^{\lambda})$ $(i = 0^{\pm}, 1)$ are isolated invariant sets satisfying (CPF2-5) with type-U adm. order. $(\lambda > \lambda^*)$

Proof of the C-PF bifurcation theorem

Outline of proof

We define

$$\Omega_\epsilon^-(M) := \{x \in Inv(N^{\lambda^*}) \mid \exists an\ \epsilon\text{-chain in } Inv(N^{\lambda^*}) \text{ from } x \text{ to } y \in M\}$$

and

$$\Omega^-(M) := \bigcap_{\epsilon>0} \Omega_\epsilon^-(M).$$

$\Rightarrow$ *We can prove*

- $\Omega^-(M(0^+)^{\lambda^*})$ *is compact and* φ^{λ^*}*-invariant.*
- *If* $\Omega^-(M(0^+)^{\lambda^*}) \cap (M(0^-)^{\lambda^*} \cup M(1)^{\lambda^*}) \neq \emptyset$, *the proof is done.*

Proof of the C-PF bifurcation theorem

Outline of proof

- *If not,* $\exists \epsilon_0 > 0$ *s.t.* $\Omega^-_{\epsilon_0}(M(0^+)^{\lambda^*}) \cap (M(0^-)^{\lambda^*} \cup M(1)^{\lambda^*}) = \emptyset$.

$\Rightarrow$ *We define*

$$R(0^+)^* := \alpha(\Omega^-_{\epsilon_0}(M(0^+)^{\lambda^*})),\ R(0^-)^* := \alpha(\Omega^-_{\epsilon_1}(M(0^-)^{\lambda^*})).$$

$\Rightarrow$ $R(0^+)^*$ *and* $R(0^-)^*$ *are isolated invariant sets satisfying*

- $M(0^\pm)^{\lambda^*} \subset R(0^\pm)^*$,
- $R(0^\pm)^*$ *and* $M(1)^{\lambda^*}$ *are mutually disjoint.*

Proof of the C-PF bifurcation theorem

Outline of proof

$A^* := \{x \in Inv(N^{\lambda^*}, \varphi^{\lambda^*}) \mid \alpha(x) \cap (R(0^+)^* \cup R(0^-)^*) = \emptyset\}$.
$\Rightarrow$ *We can prove that the collection*

$$\{A^*, R(0^-)^*, R(0^+)^*\}$$

is a Morse decomposition of $Inv(N^{\lambda^*}, \varphi^{\lambda^*})$ *with type-U admissible ordering satisfying (CPF2-5). The robustness of Morse decompositions implies the contradiction.*

Bifurcation from an nontrivial equilibrium

- Verification of equilibria

Computer assisted result

Eq. (1) has five equilibria $M(0^{\pm})^-$, $M(1^{\pm})^-$ *and* $M(2)^-$ *at* $\nu = \nu_-$ *whose Conley indices are*

$$CH_n(M(i^{\pm})^-) \cong \begin{cases} \mathbb{Z}_2 & n = i \\ 0 & n \neq i \end{cases} \quad CH_n(M(2)^-) \cong \begin{cases} \mathbb{Z}_2 & n = 2 \\ 0 & n \neq 2 \end{cases},$$

$i = 0, 1.$

Bifurcation from an nontrivial equilibrium

- Verification of equilibria

Computer assisted result

Eq. (1) has nine equilibria $M(0^{\pm})^{+}$, $M(1^{\pm})^{+}$, $M(1^{\pm\pm})^{+}$ *and* $M(2)^{+}$ *at* $\nu = \nu_{+}$ *whose Conley indices are*

$$CH_n(M(i^{\pm})^{+}) \cong \begin{cases} \mathbb{Z}_2 & n = 0 \\ 0 & n \neq 0 \end{cases} \ (i = 0, 1),$$

$$CH_n(M(1^{\pm\pm})^{+}) \cong \begin{cases} \mathbb{Z}_2 & n = 1 \\ 0 & n \neq 1 \end{cases}, \quad CH_n(M(2)^{+}) \cong \begin{cases} \mathbb{Z}_2 & n = 2 \\ 0 & n \neq 2 \end{cases}.$$

Bifurcation from the nontrivial equilibrium

Computer assisted result

We define

$$J^{\nu} := \prod_{0\leq k\leq 9} [u_k^-, u_k^+] \times \prod_{k\geq 10} \left[-\frac{1.0}{k^6}, \frac{1.0}{k^6}\right].$$

Then J^{ν} *is positively invariant for* φ^{ν}, $\nu \in [0.47607, 0.47617]$.

Table: The block J^{ν}

k	u_k^-	u_k^+
0	$-1.818042580836000 \times 10^{-1}$	$+1.818042580836000 \times 10^{-1}$
1	$-3.368056886438130 \times 10^{-1}$	$+3.368056886438130 \times 10^{-1}$
2	$-2.423873812215343 \times 10^{-1}$	$+2.423873812215343 \times 10^{-1}$
3	$-3.10477844795332 \times 10^{-2}$	$+3.10477844795332 \times 10^{-2}$
4	$-4.8171606652801 \times 10^{-3}$	$+4.8171606652801 \times 10^{-3}$
5	$-1.1011234585183 \times 10^{-3}$	$+1.1011234585183 \times 10^{-3}$
6	$-5.348203323881 \times 10^{-4}$	$+5.348203323881 \times 10^{-4}$
7	$-4.42867581713 \times 10^{-5}$	$+4.42867581713 \times 10^{-5}$
8	$-4.26077823605 \times 10^{-5}$	$+4.26077823605 \times 10^{-5}$
9	$-1.06667909570 \times 10^{-5}$	$+1.06667909570 \times 10^{-5}$

Bifurcation from an nontrivial equilibrium

- Removing isolating subneighborhoods

Computer assisted result

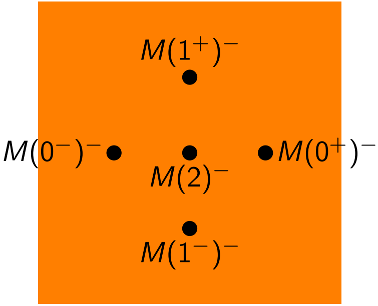
We can construct an isol. nbh. N_U for Φ $(\Phi(t, u_0, \nu) := (\varphi^\nu(t, u_0), \nu))$ s.t.

- $N_U^\nu \subset J^\nu$, $\nu \in [\nu_-, \nu_+] = [0.62163, 0.62173]$,
- $M(0^\pm)^-, M(2)^- \not\in N_U^{\nu_-}$, $M(0^\pm)^+, M(2)^+ \not\in N_U^{\nu_+}$.

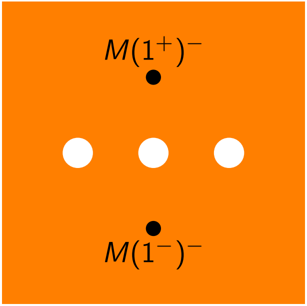
Bifurcation from an nontrivial equilibrium

- Sketch of N_U

$J^{\nu-}$

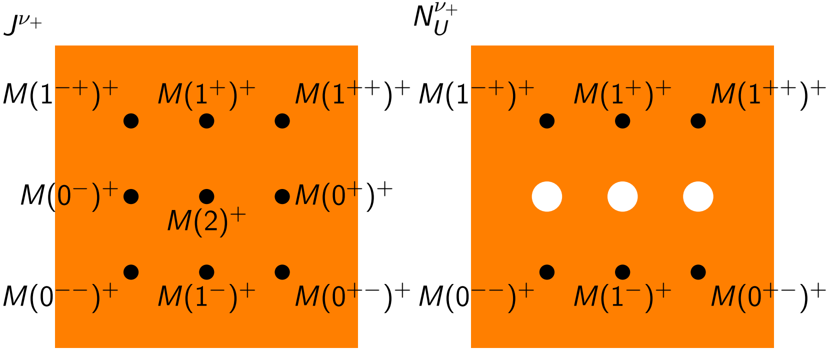


$N_U^{\nu-}$



Bifurcation from an nontrivial equilibrium

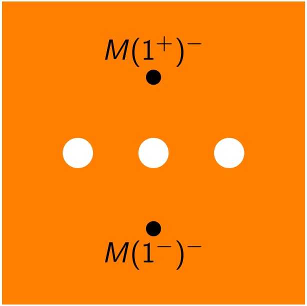
- Sketch of N_U



Bifurcation from an nontrivial equilibrium

- Sketch of global dynamics in $N_U^{\nu-}$ (by the same argument as *SIADS* (2005), 1–31).

$N_U^{\nu-}$



$$CH_n(M(1^{\pm})^-) \cong \begin{cases} \mathbb{Z}_2 & n=1 \\ 0 & n \neq 1 \end{cases}$$

$\cdot$No conn. orbits between $M(1^{\pm})^-$

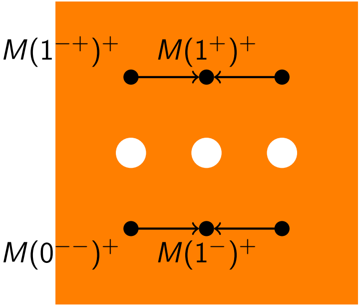
$\cdot M(1^{\pm})^-$: inv. for

$S_{\mathbb{Z}_2} : u_{2k} \mapsto u_{2k},\ u_{2k-1} \mapsto -u_{2k-1}.$

Bifurcation from an nontrivial equilibrium

- Sketch of global dynamics in $N_U^{\nu_+}$ (by the same argument as *SIADS* (2005), 1–31)

$N_U^{\nu_+}$



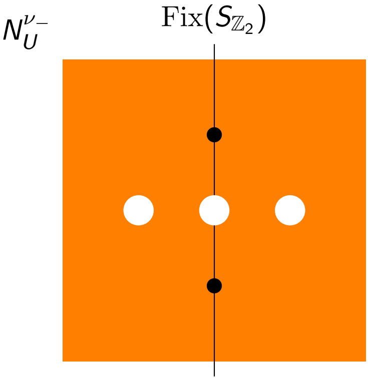
$$CH_n(M(1^{\pm})^+) \cong \begin{cases} \mathbb{Z}_2 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

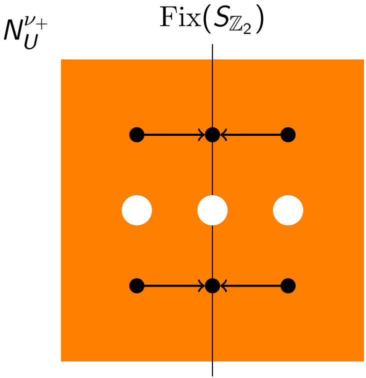
$$CH_n(M(1^{\pm\pm})^+) \cong \begin{cases} \mathbb{Z}_2 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

$\cdot M(1^{\pm})^+$: inv. for $S_{\mathbb{Z}_2}$.
$\cdot M(1^{\pm\pm})^+$: NOT inv. for $S_{\mathbb{Z}_2}$.

Bifurcation from an nontrivial equilibrium

- Sketch of global dynamics in $N_U^{\nu\pm}$ (by the same argument as *SIADS* (2005), 1–31)





Rigorous numerical methods

- Self-consistent a priori bound
  (P. Zgliczyński and K. Mischaikow, *Found. Comp. Math.*(2001), 255–288.)

H : A separable Hilbert space, $\{\varphi_i\}_{i=1,2\cdots}$: CONS on H

$$\dot{u} = F(u) \quad \text{(an evolutionary equation on) } H \tag{2}$$

"Self-consistent a priori bound" : A pair $(W, \{u_k^{\pm}\}_{k>m})$ of a compact set and a countable sequence of real numbers such that

- $W \subset span\{\varphi_1, \cdots, \varphi_m\}$
- $Z := \prod_{k>m}[u_k^-, u_k^+] \subset span\{\varphi_{m+1}, \cdots\}$
- $u \in W \times Z \Rightarrow \|u\|_H < \infty$, F is continuous on $W \times Z$.

Rigorous numerical methods

- Global dynamics
  (S. Day, Y. Hiraoka, K. Mischaikow and T. Ogawa, *SIAM J. Appl. Dyn. Sys.* (2005), 1–31)
  - Self-consistent bound
  - Unique- or non-existence of equilibria
  - Computation of the Conley index of an equilibrium
  - Connection matrix

$\Rightarrow$ Semi-conjugacy of the global attractor to an simple dynamics

- Radii polynomials
  (S. Day, J-P. Lessard and K. Mischaikow, *SIAM J. Num. Anal.* (2007), 1398–1424)

H : A separable Hilbert space, $\{\varphi_i\}_{i=1,2\cdots}$: CONS on H

$$\dot{u} = F(u) \quad : \text{an evol. eq. on } H \tag{3}$$

which is equivalent to

$$\dot{u}_k = \mu_k u_k + \sum_{n_1+\cdots+n_d=k} u_{n_1}^{p_{n_1}} \cdots u_{n_d}^{p_{n_d}}, \quad k \in \mathbb{N}, \tag{4}$$

$\mu_k \in \mathbb{R}$, $p_{n_i} \in \mathbb{N} \cup \{0\}$ satisfying $\sum_{i=1}^{d} p_{n_i} = p$.
"Radii polynomials" : A finite number of polynomials describing a priori error estimates.
Radii polynomials + contracting mapping principle $\Rightarrow$
the unique existence of an equilibrium of (4).