# Rigorous numerical verification of local dynamics around equilibria of dynamics in infinite dimensions

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Infinite dimensional dynamics . . .
Difficult to study the dynamics of concrete solutions

- Computer Assisted Proof
  (interval arithmetics, rigorous numerics)

Our goal : We briefly show

- How to verify the existence of equilibria of systems by computer assisted proof
- A criterion of the local dynamics around equilibria

Hyperbolic equilibrium.
We consider an evolutionary equation

$$\dot{u} = -Au + f(u) \tag{1}$$

on a (separable) Hilbert space $X$.

- $A$ : positive-definite self-adjoint (generally, sectorial)
- $A^{-1}$ : compact
- $f : D(A^\alpha) \to X$ : $C^1$.
- $\forall B \subset D(A^\alpha)$ : bounded $\Rightarrow f(B) \subset X$ : bounded.

## Definition

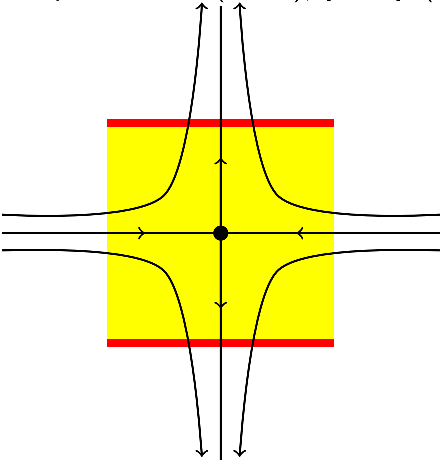
*We say an equilibrium $u^*$ for (1) is hyperbolic if, for the linearization $L := -A + df(u^*)$ at $u^*$,*

$$\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$$

*holds, where $\sigma(L)$ is the spectrum of $L$.*

## Hyperbolic equilibrium

Example : $\dot{x} = ax\ (a < 0),\ \ \dot{y} = by\ (b > 0)$



$(\sigma(L) =)\{a, b\} \cap \sqrt{-1}\mathbb{R} = \emptyset.$

$\{0\}(= S)$ is hyp. equilibrium with $\dim W^u(S) = 1$.

## The Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi] \tag{2}$$

$$u(t,-x) = -u(t,x) \text{ with per. B.C.} \tag{3}$$

Using the Fourier basis $\{\sin(k\pi x)\}_{k\geq 1}$,
$u(t,x) = -2\sum_{k\in\mathbb{N}} u_k(t)\sin(k\pi x)$ and (2)+(3) is rewritten by

$$\dot{u}_k = k^2(1-\nu k^2)u_k - k\sum_{n=1}^{k-1} u_n u_{k-n} + 2k\sum_{n=1}^{\infty} u_n u_{n+k},$$

which forms

$$\dot{u}_k = F_k(u) = d_k u_k + N_k(u), \quad k = 1, 2, \cdots .$$

Rigorous verification of equilibria

ZM-theory (Zgliczynski and Mischaikow, Found. Comp. Math. (2001) 255–288.)

Rigorous verification of equilibria of equations which form

$$\dot{u} = F(u) \Leftrightarrow \dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \cdots . \qquad (4)$$

$\Rightarrow$ Find a set $V$ (in a separable Hilbert space $X$) which forms

$$V = \prod_{k=1}^{n} [w_k^-, w_k^+] \times \prod_{k>n} \left[ -\frac{C}{k^s}, \frac{C}{k^s} \right] \quad (w_k^{\pm} \in \mathbb{R}, C > 0, s \in \mathbb{N}) \quad (5)$$

containing an equilibrium $u^*$ of (4).

## Definition

*A pair $(W, T, m)$ forms* *self-consistent bounds* *for (4) if*

- $W \subset X_m = P_m X$ *(m-dim. subsp.) : compact, convex*
- $T = \prod_{k>m} \left[-\frac{C}{k^s}, \frac{C}{k^s}\right] \subset Y_m = (I_X - P_m)X$ $(w_k^{\pm} \in \mathbb{R}, C > 0, s \in \mathbb{N})$
- $V = W \oplus T \subset X$
- *$F$ is continuous on $V$,* $\sup_{u \in V} \|F(u)\| < \infty$*.*

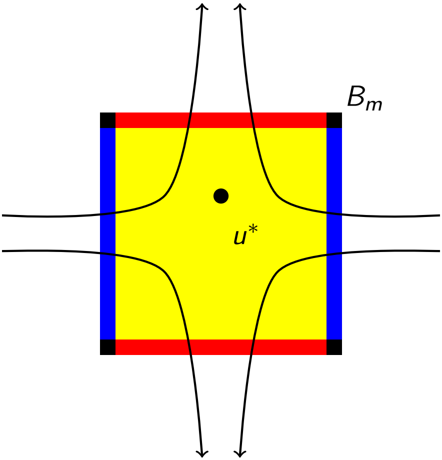
- For $u \in V$, $\left|\sum_{k>m} u_k\right| \leq C \int_m^\infty x^{-s} dx = C/(s-1)m^{s-1}$.
- $\Rightarrow \sum_{k>m} u_k \in [-C/(s-1)m^{s-1}, +C/(s-1)m^{s-1}]$.
- Nonlinear term $N_i(u)$ on $V$ is included in an interval $[\delta_i^-, \delta_i^+]$.

For $\dot{u}_i = d_i u_i + N_i(u)$, we assume that

- $d_k \in \mathbb{R} \setminus \{0\}$ for all $k$ (not essential),
- $d_k < 0$ for $k > m$,
- $N_k(u) \subset [\delta_k^-, \delta_k^+]$ for $u \in V$, $k \geq 1$,
- $u_k = -C/k^s \quad \Rightarrow \quad F_k(u) > 0$ for $k > m$,
- $u_k = +C/k^s \quad \Rightarrow \quad F_k(u) < 0$ for $k > m$,

i.e. Vector Field is contracting for $k > m$ (entrance condition).
$\Rightarrow B_m := \sum_{i=1}^m [b_i^-, b_i^+]$, where

$$[b_i^-, b_i^+] := \left[-\frac{\delta_i^+}{d_i}, -\frac{\delta_i^-}{d_i}\right] \qquad \text{if } d_i > 0,$$
$$[b_i^-, b_i^+] := \left[-\frac{\delta_i^-}{d_i}, -\frac{\delta_i^+}{d_i}\right] \qquad \text{if } d_i < 0.$$




the entrance ($d_i < 0$)

$= $ *vertical lines*

the exit ($d_i > 0$)

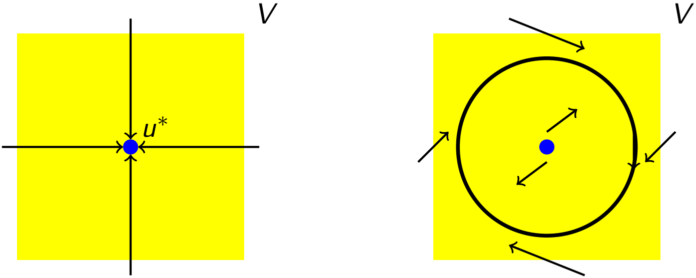
$= $ *horizontal lines* + vertices

The Conley index theory

$\Rightarrow \exists u^* \in B_m \oplus T$ s.t. $F(u) = 0$

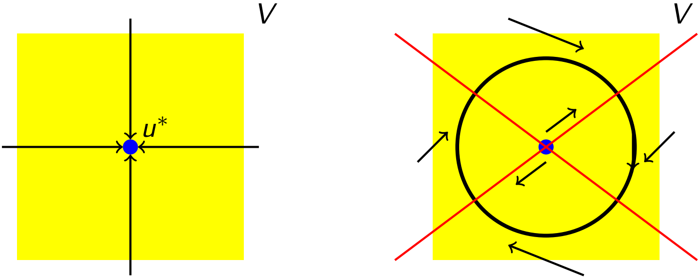
(if $B_m \subset W$)

## Local dynamics around equilibria




Even if an equilibrium $u^*$ is unique in $V$, $\{u^*\} \neq Inv(V)$ in general.
$\Rightarrow$ Stability of $\{u^*\} \neq$ Stability of $Inv(V)$

If dynamics in $V$ is gradient-like . . .




All recurrent invariant sets (e.g. periodic orbits) are excluded.
$\Rightarrow$ Precise structure of $Inv(V)$

Hyperbolicity verification theorem.
We consider an evolutionary equation on $X$

$$\dot{u} = F(u) \Leftrightarrow \dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \cdots . \tag{6}$$

- $u = (u_1, u_2, \cdots) \in X, \quad V \subset X$ which forms (5).
- $\partial F_i / \partial u_j \in C(V, \mathbb{R})$,
  $\sum_{j \geq 1} \max_{u \in V} |(\partial F_i / \partial u_j)(u)| \sup_{x,y \in V} |x_j - y_j| < \infty$.

Hyperbolicity verification theorem.

Our requirement . . .

- Construction of Lyapunov function on $V$

$\Rightarrow$ NO recurrent invariant sets in $V$.

**Assumption**

$$\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} = \emptyset \tag{7}$$

*holds for all $u \in V$.*

## Theorem (M.)

*Let $u^*$ be an equilibrium of (6) in $V$. If (7) holds and*

$$\inf_{i\in\mathbb{N}}\left(|d_i| - \sum_{j\geq 1}\sup_{u\in V}\left|\frac{\partial N_i}{\partial x_j}(u)\right|\right) = \delta_1 > 0, \tag{8}$$

$$\inf_{i\in\mathbb{N}}\left(|d_i| - \sum_{j\geq 1}\sup_{u\in V}\left|\frac{\partial N_j}{\partial x_i}(u)\right|\right) = \delta_2 > 0 \tag{9}$$

$$m := \text{ the number of } d_i \text{ with positive real part} \tag{10}$$

*hold, then $Inv(V) = \{u^*\}$. $u^*$ is hyperbolic (for (6)) with* $\dim W^u(u^*) = m$.

## Outline of proof

*Construct a Lyapunov function of the form*

$$L(u) := -\sum_{i\geq 1} sign(d_i)\cdot (u_i - u_i^*)^2,$$

*where $u^*$ is an equilibrium of $\dot{u} = F(u)$ in $V$.*

- *Define $G(u) := \frac{dL}{dt}(u)$.*
- *Assumption $\Rightarrow$ the 2nd variation of $G$ is strictly negative definite.*
- *$\Rightarrow G(u)\mid_{t=0} \leq 0$ for $u \in V$ and $G(u) = 0$ iff $u = u^*$. (i.e. $L$ is a Lyapunov function.)*

Hyperbolicity ($\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} = \emptyset$) ...

- (A part of) Nakao's theory ... Eigenvalue exclusion
  Ref. : M.T. Nakao, K. Hashimoto and Y. Watanabe, *Computing*, **75**(2005), 1–14.
- A fact of parabolic evolutionary equations

$$\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} \subset \{iy \mid y \in [-a, a]\}, \quad a > 0.$$

Ex. : the Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi],$$
$$u(t,x) = -u(t,-x), u(t, x+2\pi) = u(t,x).$$

## Computer assisted result

$$\exists u^* \in V = \{u_{32}\} + \prod_{k=1}^{32} [-w_k, w_k] \times \prod_{k>32} \left[-\frac{C}{k^s}, \frac{C}{k^s}\right]$$

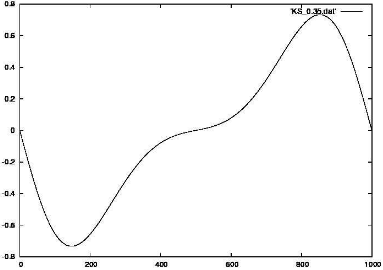
*for* $\nu = 0.35$*, as the next slide.*

- $Inv(V) = \{u^*\}$.
- $u^*$ *is hyperbolic with* $\dim W^u(u^*) = 0$.

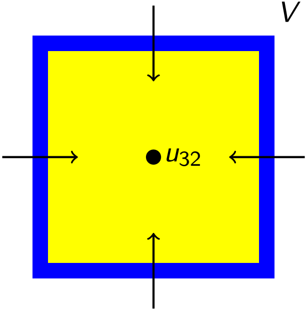
Equilibrium $u_*$ of $u_t = -0.35u_{xxxx} - u_{xx} + 2uu_x$ in $[-\pi, \pi]$

$$C = 2040.74723, \;\; s = 6,$$

$$\sup_{1 \leq k \leq 32} w_k \leq 3.59468318 \times 10^{-5}.$$



Phase portrait around $u^*$ in $L^2_{per}(-\pi,\pi)$

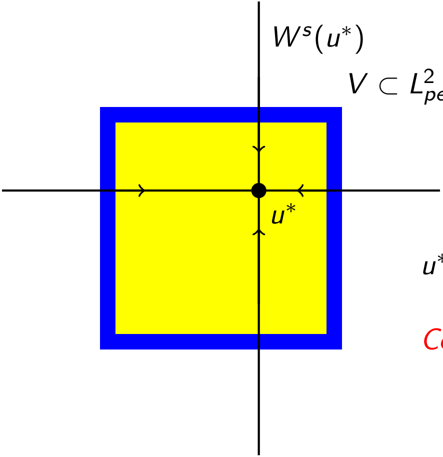



$V \subset L^2_{per}(-\pi,\pi)$

the entrance ($d_i < 0$)
$=$ *whole boundary* of $V$

$u_{32}$ : approx. sol.

Phase portrait around $u^*$ in $L^2_{per}(-\pi,\pi)$




the entrance ($d_i < 0$)
$=$ *whole boundary* of $V$

$u^* : -0.35u_{xxxx} - u_{xx} + 2uu_x = 0.$

*Covering relations*

$\Rightarrow$ More precise description of $W^u(u^*)$ and $W^s(u^*)$.