Rigorous numerics for certain solutions of PDEs existence, uniqueness, dynamics

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Workshop on nonlinear equations
@ Hirosaki Univ. 2011 12.22

Well-known results for “dissipative” dynamical systems...

- Generation of dynamical system
- Existence of the “global attractor”

$\Rightarrow$

- The inner structure of the global attractor ... ??
- Complex behavior

$\Rightarrow$ COMPUTER ASSISTED PROOF

1st step : verification of stationary solutions

2nd step : periodic solutions, traveling waves, etc.

A topological approach

- Conley-type index

Rigorous numerics

- $\sqrt{2} \approx 1.41421356$: not exact
- $\sqrt{2} - 1.41421356 \neq 0$: ROUNDING ERROR
- $\sqrt{2} \in [1.41421356, 1.41421357]$: mathematically exact

$a + c$ (Floating point number arithmetic)
$\Rightarrow [a, b] + [c, d] = [a + c, b + d]$ etc. (*interval arithmetic*)

- $S = \sum_{n=1}^{\infty} a_n$: not computable *rigorously* in general
- $S_N = \sum_{n=1}^{N} a_n$: computable but $\neq S$
- $S_N - S$: TRUNCATION ERROR

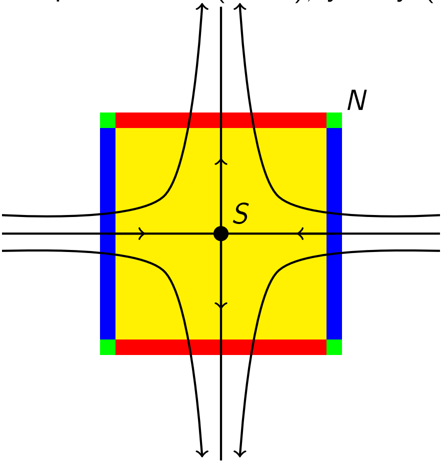
Assuming $a_n \in [-N^{-2}, N^{-2}]$ for $n > N$ ...
$\Rightarrow S \in S_N + [-N^{-1}, N^{-1}]$: computable and *mathematically rigorous*

The Conley-type index :

- X : a complete metric space
- π : semiflow on X ($x \cdot t := \pi(t, x)$, $x \cdot (t+s) = (x \cdot t) \cdot s$)
- $N \subset X$: closed
- $Inv_\pi(N) := \{x \in X \mid \exists \sigma : (-\infty, \omega_x) \to X$: solution s.t. $\sigma(0) = x,\ \sigma((-\infty, \omega_x)) \subset N\}$
- N : isolating neighborhood $\Leftrightarrow Inv_\pi(N) \subset int(N)$
- $S \subset X$: isolated invariant set $\Leftrightarrow \exists N$ isolating neighborhood s.t. $S = Inv_\pi(N)$.

An Isolating block.

Example : $\dot{x} = ax\ (a < 0),\ \ \dot{y} = by\ (b > 0)$



vertical lines : entrance

N^+ := the exit

= *horizontal lines* + *vertices*

(N^+ is closed.)

$\Rightarrow$ N is an isolating block.

The pair (N, N^+) is called an "index pair".

Definition

- *S : isolated invariant set*
- *(N_1, N_2) : index pair of S in an isol. nbh. N where "π is asymptotically compact"*

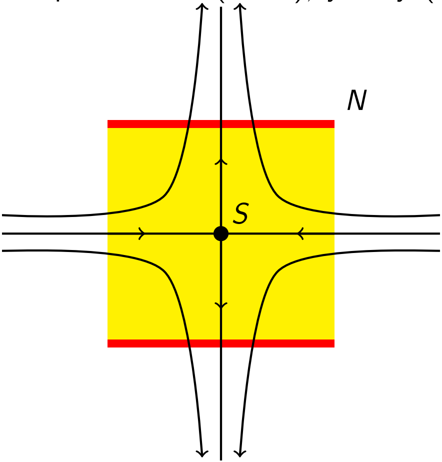
$\Rightarrow$ the Conley-type index *is the homology group*

$$CH_*(S, \pi) = H_*(N_1/N_2, [N_2]).$$

- π : asymptotically compact on $N \Rightarrow S = Inv(N)$: compact
- $CH_*(S, \pi)$ does not depend on the choice of index pairs of S.
- $CH_*(S, \pi) \ncong 0 \Rightarrow S \neq \emptyset$.

The Conley-type index

Example : $\dot{x} = ax \ (a < 0), \ \dot{y} = by \ (b > 0)$



(N, N^+) : index pair

$$CH_n(S) \cong \begin{cases} R & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

We consider the strongly damped wave equation on a Hilbert space H:

$$u_{tt} = \alpha \Delta u_t + \Delta u + f(u) \text{ in } \Omega \tag{1}$$

with

$$u = 0 \text{ on } \partial\Omega$$

$\alpha > 0$: a constant.
$\Omega \subset \mathbb{R}^n$: a bounded domain
$f : U(\subset X, \text{ open}) \to Y$: $\sup_{|u|\to\infty}(f(u)/u) \leq 0$
$X \subset Y \subset H$: Hilbert spaces

The strongly damped wave equation (1) can be rewritten by

$$\dot{V} = DU + G(U), \tag{2}$$

$V = (u, v)^T$, $D = \begin{pmatrix} 0 & I \\ -A & -\alpha A \end{pmatrix}$, $G(U) = (0, f(u))^T$.
$A = -\Delta$. $D(A)$ includes the boundary condition.

Now we transform (u, v) into (ϕ, ψ), where

$$\phi = u, \psi = \gamma u + \delta v.$$

Then, (2) is equivalent to

$$\dot{U} = K(U) := CU + F(U), \tag{3}$$

$$U = (\phi, \psi)^T,\ C = \begin{pmatrix} -\gamma\delta^{-1} & \delta^{-1} \\ -\gamma^2\delta^{-1} + (\alpha\gamma - \delta)A & \gamma\delta^{-1} - \alpha A \end{pmatrix},$$

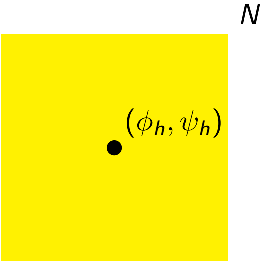
$$F(U) = (0, \delta f(\phi))^T.$$

We go back to the following system in $X = H_0^1(\Omega) \times H_0^1(\Omega)$:

$$\begin{cases} \dot{p} = P_h K(p,q) \\ \dot{q} = (I - P_h)K(p,q) \end{cases}, \quad K(p,q) = K(U) = CU + F(U).$$

- X_h : Finite Element Subspace, P_h:ortho. proj. onto X_h
- $p = P_h U$, $q = (I - P_h)U$
- $N := N_1 \times N_2 \subset X$
- $N_1 = \prod_{i=1}^{\dim X_h}[a_i^-, a_i^+] \subset X_h$, (ϕ_h, ψ_h) (approximation) $\in N_1$
- $N_2 = \{(q_1, q_2) \in (I - P_h)X \mid (\|q_1\|_{H_0^1}^2 + C_0\|q_2\|_{L^2}^2)^{1/2} \leq M\}$
- π : semiflow for (3)
- π_q : semiflow for $\dot{p} = P_h F(p,q)$, $q \in N_2$

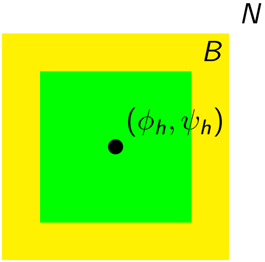
Scenario of verification



(ϕ_h, ψ_h) : approx. stationary sol.

N : given set

Scenario of verification

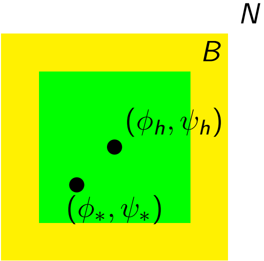


(ϕ_h, ψ_h) : approx. stationary sol.

N : given set

B : candidate of isolating block with $CH_*(Inv(B)) \not\cong 0$

Scenario of verification



(ϕ_h, ψ_h) : approx. stationary sol.
N : given set
B : candidate of isolating block
with $CH_*(Inv(B)) \not\cong 0$

If $B \subset N \Rightarrow \exists (\phi_*, \psi_*) \in int(B)$: stationary sol. of original PDE

Next step : Lifting

- Restriction of the dynamics for the tail
  $\Rightarrow$ We know the infinite dimensional dynamics from the finite dimensional one *AUTOMATICALLY*.
- As for the Conley index, the exit is essential.

Fact

R.Mañé : "The dimension of the global attractor (for (1)) is finite."

Next step : Lifting

- Restriction of the dynamics for the tail
  $\Rightarrow$ We know the infinite dimensional dynamics from the finite dimensional one *AUTOMATICALLY*.
- As for the Conley index, the exit is essential.
  $\Rightarrow$ NO EXIT in the tail.

Entrance Condition

We assume that

$$\frac{d}{dt}(\|q_1\|^2_{H^1_0} + C_0\|q_2\|^2_{L^2}) < 0 \qquad (4)$$

for all $(\phi, \psi) \in N$ *with* $(\|q_1\|^2_{H^1_0} + C_0\|q_2\|^2_{L^2})^{1/2} = (\|(I - P_h)\phi\|^2_{H^1_0} + C_0\|(I - P_h)\psi\|^2_{L^2})^{1/2} = M$.

Entrance Condition

We assume that

$$\frac{d}{dt}(\|q_1\|^2_{H^1_0} + C_0\|q_2\|^2_{L^2}) < 0 \qquad (5)$$

for all $(\phi,\psi) \in N$ *with* $(\|q_1\|^2_{H^1_0} + C_0\|q_2\|^2_{L^2})^{1/2} = (\|(I-P_h)\phi\|^2_{H^1_0} + C_0\|(I-P_h)\psi\|^2_{L^2})^{1/2} = M$.

Theorem (M.)

Let $N = N_1 \times N_2$ *be a strongly* π*-admissible closed set in* X*. We assume that (5) holds.*

1. *If* N_1 *is an isol. block for* π_q *for all* $q \in N_2$*, then so is* N *for* π*.*
2. $CH_*(Inv(N),\pi) = CH_*(Inv(N_1),\pi_q)$ *for all* q*.*

We set

$$\gamma := 2/\alpha, \quad \delta := 2, \quad C_0 = C_0(\alpha, h) \approx 1$$

so that we translate (5) into “an inequality”.

Proposition (M.)

Condition (5) holds if the following inequality is satisfied:

$$\gamma_*(h)(2\lambda \sup_{u\in N} \|f(u)\|_{L^2} + F_N) < M, \tag{6}$$

where

$$\gamma_*(h) = 2(0.5 - 4C_0(Ch)^2\alpha^{-2})C_0 Ch,$$
$$F_N = F_N(\gamma, \delta, \phi, \psi) \ : \ \textit{computable},$$

C : a constant depending on the choice of the basis on X_h. h : the mesh size.

Main theorem

Theorem (M.)

Let

$$\begin{cases} N_1 := \{(\phi_h, \psi_h)\} + \prod_{k=1}^{\dim X_h}[a_k^-, a_k^+], \\ N_2 := [M] = \{q \in (I - P_h)X \mid \|q_1\|_{H_0^1}^2 + C_0\|q_2\|_{L^2}^2 \leq M^2\}. \end{cases}$$

If there exists an isolating block B in N_1 such that

- $CH_n(Inv(B), \pi_q) \cong \begin{cases} R & n = \ell \text{ for some } \ell \\ 0 & \text{otherwise} \end{cases}, \forall q \in N_2$
- $\gamma_*(h)(2\lambda \sup_{u \in N} \|f(u)\|_{L^2} + F_N) < M$

hold, then there exists an equilibrium of (3) in $N := N_1 \times N_2$.

Next step : *Construction of ISOLATING BLOCK*

$$\dot{U} = K(U) \Rightarrow \begin{cases} \dot{p} = P_h K(U) & \Leftarrow \text{Conley-type index} \\ \dot{q} = (I - P_h) K(U) & \Leftarrow \text{Entrance condition} \end{cases}$$

$\Rightarrow$ Consider the dynamics of the lower term around u_h.
$\Rightarrow$ Transform the original system to the perturbed diagonalized system.

Original System $\Rightarrow$ Perturbed Diagonalized System around (ϕ_h, ψ_h). A solution $U = (\phi, \psi)^T$ satisfies

$$
\begin{aligned}
(\phi_t, v) &= (-\gamma\delta^{-1}\phi + \delta^{-1}\psi, v) \\
(\psi_t, w) &= (-\gamma^2\delta^{-1}\phi + \gamma\delta^{-1}\psi, w) - ((\alpha\gamma - \delta)\nabla\phi + \alpha\nabla\psi, \nabla w) \\
&\quad + \delta(f(\phi), w), \text{ for all } (v, w) \in X.
\end{aligned}
$$

$\Rightarrow$ Finite dimensional projection:

$$
\begin{aligned}
(\phi_t, v_h) &= (-\gamma\delta^{-1}\phi + \delta^{-1}\psi, v_h) \\
(\psi_t, w_h) &= (-\gamma^2\delta^{-1}\phi + \gamma\delta^{-1}\psi, w_h) - ((\alpha\gamma - \delta)\nabla\phi + \alpha\nabla\psi, \nabla w_h) \\
&\quad + \delta(f(\phi), w_h), \text{ for all } (v_h, w_h) \in X_h.
\end{aligned}
$$

If we set $(\tilde{\phi}, \tilde{\psi}) := (\phi - \phi_h, \psi - \psi_h)$, it must satisfy

$$(\nabla(\tilde{\phi}_h)_t, \nabla v_h) = (-\gamma\delta^{-1}\nabla\tilde{\phi} + \delta^{-1}\nabla\tilde{\psi}, \nabla v_h)$$
$$((\tilde{\psi}_h)_t, w_h) \in (-\gamma^2\delta^{-1}\tilde{\phi} + \gamma\delta^{-1}\tilde{\psi}, w_h) - ((\alpha\gamma - \delta)\nabla\tilde{\phi} + \alpha\nabla\tilde{\psi}, \nabla w_h)$$
$$+ \delta(f(\phi), w_h), \text{ for all } (v_h, w_h) \in X_h.$$

$\Rightarrow$ ("diagonalization")

$$\dot{Y}_h \in \Lambda Y_h + Q\Phi^{-1}(\epsilon_1(w), \ldots, \epsilon_{\dim X_h}(w))^T.$$
$$(\dot{y}_i \in \lambda_i y_i + \tilde{\epsilon}_i(\tilde{y}), \; i = 1, \ldots, \dim X_h)$$

- $\Lambda = diag(\lambda_1, \ldots, \lambda_{\dim X_h})$.
- Q, Φ : nonsingular linear transformations.
- $\tilde{\epsilon}_i(\tilde{y}) = o(|y|^2)$ (as $|y| \to 0$) : the i-th bound of error terms.

Set $\tilde{\epsilon}_i(\tilde{y}) := [\delta_i^-, \delta_i^+]$. $\dot{y}_i$ must satisfy the following relation:

$$\lambda_i \left(y_i + \frac{\delta_i^-}{\lambda_i} \right) < \dot{y}_i < \lambda_i \left(y_i + \frac{\delta_i^+}{\lambda_i} \right).$$

$\Rightarrow$ the candidate of our isolating neighborhood is as follows:

$$\begin{cases} \tilde{W}_i^{(k)} := \left[-\frac{\delta_i^+}{\lambda_i}, -\frac{\delta_i^-}{\lambda_i} \right], & \text{if } \lambda_i > 0, \\ \tilde{W}_i^{(k)} := \left[-\frac{\delta_i^-}{\lambda_i}, -\frac{\delta_i^+}{\lambda_i} \right], & \text{if } \lambda_i < 0. \end{cases} \tag{7}$$

Remark

Our isolating block is $\left(\{(\phi_h, \psi_h)\} + \prod_{i=1}^{\dim X_h} Q^{-1} \tilde{W}_i^{(k)} \right) \times [M]$.
To compute the CR-index, we just count the number of $i \in \{1, \ldots, \dim X_h\}$ *such that* $\lambda_i > 0$.

Hyperbolic equilibrium.
We consider an evolutionary equation

$$\dot{u} = -Au + f(u) \tag{8}$$

on a (separable) Hilbert space X.

- A : positive-definite self-adjoint (generally, sectorial)
- A^{-1} : compact
- $f : D(A^\alpha) \to X$: C^1.
- $\forall B \subset D(A^\alpha)$: bounded $\Rightarrow f(B) \subset X$: bounded.

Definition

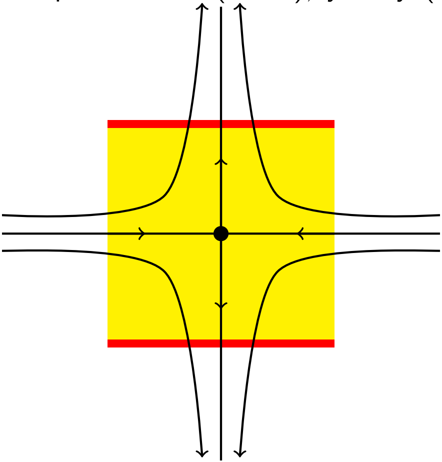
We say an equilibrium $\bar{u}$ for (8) is hyperbolic if, for the linearization $L := -A + df(\bar{u})$ at $\bar{u}$,

$$\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$$

holds, where $\sigma(L)$ is the spectrum of L.

Hyperbolic equilibrium

Example : $\dot{x} = ax \ (a < 0), \ \dot{y} = by \ (b > 0)$



$$(\sigma(L) =)\{a, b\} \cap \sqrt{-1}\mathbb{R} = \emptyset.$$

$\{0\}(= S)$ is hyp. equilibrium with $\dim W^u(S) = 1$.

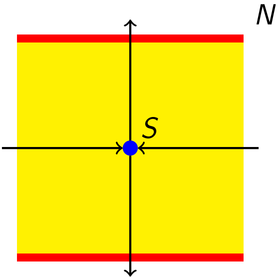
Preceding results for verifying equilibria of PDEs $\cdots$

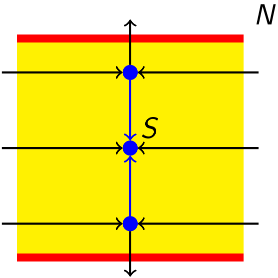
- Existence
- Stability (e.g. Conley index)
- Uniqueness

of equilibria in a subset N of a Banach space.

(Zgliczyński, Gameiro, Lessard, Mischaikow, ...)
(Nakao et. al., Oishi et. al.)

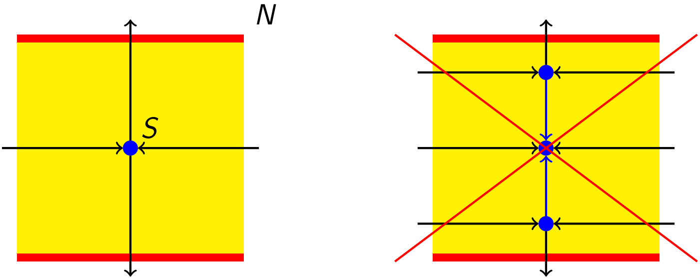
"Existence of equilibria"
(Schauder's fixed point theorem, Conley index, Brouwer's degree)





At least one equilibrium in N. Which is the true situation ?

"Existence + Local uniqueness"
(Contraction mapping principle)

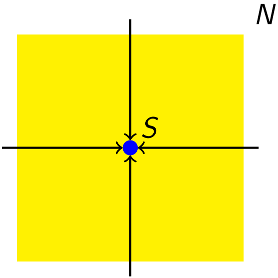


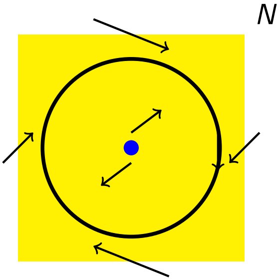
Equilibrium is unique in N. NOT the case of the right figure.

As an invariant set of dynamical systems . . .

- Gradient dynamics : Inv. sets = equilibria + connecting orbits
- General dynamics : + "recurrent" inv. sets (e.g. periodic orbits, chaotic inv. sets)

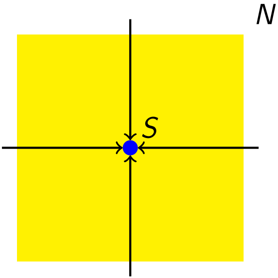
Gradient dynamics vs. General dynamics

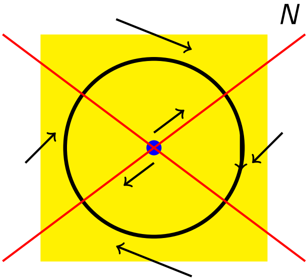




Even if an equilibrium $\bar{u}$ is unique in N, $\{\bar{u}\} \neq Inv(N)$ in general.
$\Rightarrow$ Stability of $\{\bar{u}\} \neq$ Stability of $Inv(N)$

If dynamics in N is gradient-like . . .





All recurrent invariant sets (e.g. periodic orbits) are excluded.
$\Rightarrow$ Precise structure of $Inv(N)$

Our requirements ...

- Local uniqueness of equilibrium
- Hyperbolicity of equilibrium
- Construction of Lyapunov function on N
  (so that dynamics is gradient-like in N)

Hyperbolicity verification theorem.

Two types of verification theorems
depending on rigorous numerical verification method.

1. Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) := d_i u_i + N_i(u), \quad i = 1, 2, \cdots .$$

2. Other form.
(In case that u is not always expanded by eigenfunctions)

The Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi] \tag{9}$$
$$u(t,-x) = -u(t,x) \text{ with per. B.C.} \tag{10}$$

Using the Fourier basis $\{\sin(k\pi x)\}_{k\geq 1}$,
$u(t,x) = -2\sum_{k\in\mathbb{N}} u_k \sin(k\pi x)$ and (9)+(10) is rewritten by

$$\dot{u}_k = k^2(1-\nu k^2)u_k - k\sum_{n=1}^{k-1} u_n u_{k-n} + 2k\sum_{n=1}^{\infty} u_n u_{n+k},$$

which forms

$$\dot{u}_k = F_k(u) = d_k u_k + N_k(u), \quad k = 1, 2, \cdots .$$

Rigorous verification of equilibria

ZM-theory (Zgliczynski and Mischaikow, Found. Comp. Math. (2001) 255–288.)

Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \cdots . \tag{11}$$

$\Rightarrow$ Find a set V (in a Hilbert space X) which forms

$$V = \prod_{k=1}^{n} [w_k^-, w_k^+] \times \prod_{k>n} \left[-\frac{C}{k^s}, \frac{C}{k^s} \right] \quad (w_k^\pm \in \mathbb{R}, C > 0, s \in \mathbb{N}) \tag{12}$$

containing an equilibrium u^* of (11).
(*Tool* : Conley-type index or mapping degree)

Hyperbolicity verification theorem.

We consider an evolutionary equation on X:

$$\dot{u} = F(u) \Leftrightarrow \dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \cdots . \tag{13}$$

- $d_i < 0$ for all sufficiently large i.
- $u = (u_1, u_2, \cdots) \in X, \quad V \subset X$ which forms (12).
- $\partial F_i / \partial u_j \in C(V, \mathbb{R})$,
  $\sum_{j \geq 1} \max_{u \in V} |(\partial F_i / \partial u_j)(u)| \sup_{x,y \in V} |x_j - y_j| < \infty$.

Theorem (Zgliczyński)

If, for all $i \in \mathbb{N}$,

$$\inf_{u \in V} \left| \frac{\partial F_i}{\partial x_i}(u) \right| > \sum_{j \neq i} \sup_{u \in V} \left| \frac{\partial F_i}{\partial x_j}(u) \right|$$

holds, then $F(x) = 0$ has at most one solution in V.

Assumption

$$\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} = \emptyset \tag{14}$$

holds for all $u \in V$.

Theorem (M.)

Let u^* be an equilibrium of (13) in V. If (14) holds and

$$\inf_{i\in\mathbb{N}}[|d_i| - \sum_{j\geq 1} \sup_{u\in V} \left|\frac{\partial N_i}{\partial x_j}(u)\right|] = \delta_1 > 0, \tag{15}$$

$$\inf_{i\in\mathbb{N}}[|d_i| - \sum_{j\geq 1} \sup_{u\in V} \left|\frac{\partial N_j}{\partial x_i}(u)\right|] = \delta_2 > 0 \tag{16}$$

$$m := \text{ the number of } d_i \text{ with positive real part} \tag{17}$$

hold, then $Inv(V) = \{u^*\}$. u^* is hyperbolic (for (13)) with $\dim W^u(\{u^*\}) = m$.

Outline of proof

Construct a Lyapunov function of the form

$$L(u) := -\sum_{i\geq 1} sign(d_i)\cdot (u_i - u_i^*)^2,$$

where u^* is the unique equilibrium of $\dot{u} = F(u)$ in V.

- *Define $G(u) := \frac{dL}{dt}(u)$.*

Outline of proof

$$\frac{1}{2}\frac{\partial G}{\partial u_i} = -sign(d_i)\{2d_i(u_i - u_i^*) + (1+\delta_{ij})\sum_{j\geq 1}\frac{\partial N_i(c_{ij})}{\partial u_j}(u_j - u_j^*)\}$$

$$\frac{1}{2}\frac{\partial^2 G}{\partial u_i \partial u_j} = -sign(d_i)\left\{2d_i\delta_{ij} + \left(\frac{\partial N_i(c_{ij})}{\partial u_j} + \frac{\partial N_j(c_{ji})}{\partial u_i}\right)\right\}$$

By (15) and (16),

- *u is the critical point of $G \Leftrightarrow u = u^*$.*
- *G is strictly negative-definite. $G(u) = 0$ iff $u = u^*$ (in V).*
- *$\dot{G} \leq 0$. Thus L is a Lyapunov function in V.*

In case of FEM...
Cannot expand u by *eigenfunctions*.

$$u = \sum_{i=1}^{n} u_i \varphi_i + \varphi_\perp, \quad \varphi_\perp \in S_h^\perp$$

- Behavior of Lyapunov function
  $\Rightarrow$ perturbed diagonal system + *entrance condition* (M.)
- No information about spectrum in the tail term ... ?
  $\Rightarrow$ *the Conley-type index* (M.)

Rigorous numerical verification of hyperbolic equilibria

Existence of equilibria

ZM-theory
(or "FEM + the Conley-type index" (M.)).

Lyapunov function

Main theorem which is shown before.

Eigenvalue exclusion (= hyperbolicity)

Nakao's theory.
Ref. : M.T. Nakao, K. Hashimoto and Y. Watanabe, *Computing*, **75**(2005), 1–14.

Ex. : the Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi],$$
$$u(t,x) = -u(t,-x), u(t, x+2\pi) = u(t,x).$$

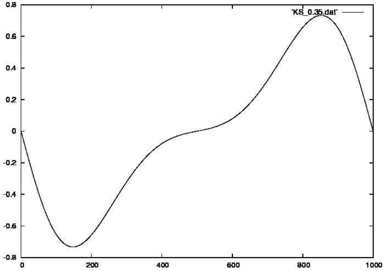
Remark

$$u^* \in \{u_{32}\} + \prod_{k=1}^{32} [-w_k, w_k] \times \prod_{k>32} \left[-\frac{C}{k^s}, \frac{C}{k^s} \right]$$

Equilibrium u_* of $u_t = -0.35u_{xxxx} - u_{xx} + 2uu_x$ in $[-\pi, \pi]$

$$C = 2040.74723,\ \ s = 6,\ \ \sup_{1\le k\le 32} w_k \le 3.59468318 \times 10^{-5}$$

$$\dim W^u(u^*) = 0.$$



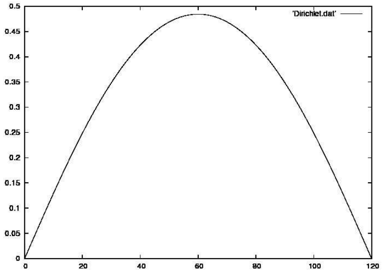
Ex. : Strongly damped wave equation (0-Dirichlet)

$$
\begin{aligned}
&u_{tt} = u_{txx} + u_{xx} + \lambda(u - u^3) \text{ in } I = (0,1)\\
&u(t,0) = u(t,1) = 0.
\end{aligned}
$$

Equilibrium u_* of $u_{tt} = u_{txx} + u_{xx} + 12(u - u^3)$ in $I = (0, 1)$

$$\sup_{1\leq k\leq 60} w_k \leq 4.11387465 \times 10^{-4}, \ M = 4.0619661 \times 10^{-4}$$

$\dim W^u(u^*) = 0.$



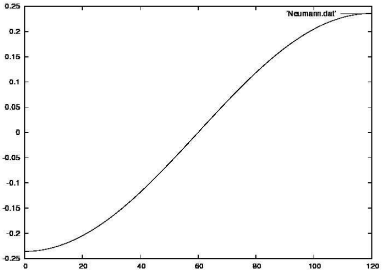
Ex. : Strongly damped wave equation (0-Neumann)

$$u_{tt} = u_{txx} + u_{xx} + \lambda(u - u^3) \text{ in } I = (0,1)$$
$$u_x(t,0) = u_x(t,1) = 0.$$

Equilibrium u_* of $u_{tt} = u_{txx} + u_{xx} + 10.3(u - u^3)$ in $I = (0,1)$

$$\sup_{1\leq k\leq 60} w_k \leq 2.2024165 \times 10^{-4}, \ \ M = 2.0053294 \times 10^{-4}$$

$\dim W^u(u^*) = 1.$



Future works:

- Connecting orbits between stationary solutions of PDEs.
  - Hyperbolicity + covering relation, cone condition
    $\Rightarrow$ description of stable and unstable manifolds
  - Known numerical verification methods
    $\Rightarrow$ ODE-like methods to PDEs.
- Hyperbolic periodic orbits of PDEs.
  - Rigorous computation of Poincaré maps (Zgliczyński)
  - Topological tools (covering relation + discrete Conley index)
    $\Rightarrow$ computation of $\dim W^u(\gamma)$ (M.)

$\Rightarrow$ GLOBAL DYNAMICS

$\Rightarrow$ Further applications (Bifurcation problems)