

# Rigorous numerics for certain solutions of PDEs existence, uniqueness, dynamics

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Well-known results for “dissipative” dynamical systems...

- Generation of dynamical system
- Existence of the “global attractor”

⇒

- The inner structure of the global attractor ... ??
- Complex behavior

⇒ COMPUTER ASSISTED PROOF

1st step : verification of stationary solutions

2nd step : periodic solutions, traveling waves, etc.

A topological approach

- Conley-type index

## 1 Preliminaries

- "Rigorous" numerics
- The Conley-type index

## 2 Validated computation

- Lifting
- The tail term
- The lower term

## 3 Local dynamics around equilibria

- Hyperbolic equilibrium
- Preceding results and Our requirements
- Hyperbolicity verification theorem

## 4 Application

- Rigorous numerical verification methods
- Examples

## Rigorous numerics

- $\sqrt{2} \approx 1.41421356$  : not exact
- $\sqrt{2} - 1.41421356 \neq 0$  : **ROUNDING ERROR**
- $\sqrt{2} \in [1.41421356, 1.41421357]$  : mathematically exact

$a + c$  (Floating point number arithmetic)

$\Rightarrow [a, b] + [c, d] = [a + c, b + d]$  etc. (*interval arithmetic*)

- $S = \sum_{n=1}^{\infty} a_n$  : not computable *rigorously* in general
- $S_N = \sum_{n=1}^N a_n$  : computable but  $\neq S$
- $S_N - S$  : **TRUNCATION ERROR**

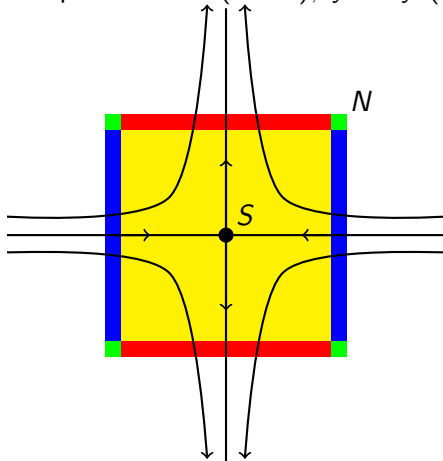
Assuming  $a_n \in [-N^{-2}, N^{-2}]$  for  $n > N \dots$

$\Rightarrow S \in S_N + [-N^{-1}, N^{-1}]$  : computable and *mathematically rigorous*

## The Conley-type index :

- $X$  : a complete metric space
- $\pi$  : semiflow on  $X$  ( $x \cdot t := \pi(t, x)$ ,  $x \cdot (t + s) = (x \cdot t) \cdot s$ )
- $N \subset X$  : closed
- $Inv_{\pi}(N) := \{x \in X \mid \exists \sigma : (-\infty, \omega_x) \rightarrow X : \text{solution}$   
s.t.  $\sigma(0) = x, \sigma((-\infty, \omega_x)) \subset N\}$
- $N$  : **isolating neighborhood**  $\Leftrightarrow Inv_{\pi}(N) \subset int(N)$
- $S \subset X$  : **isolated invariant set**  $\Leftrightarrow \exists N$  isolating neighborhood  
s.t.  $S = Inv_{\pi}(N)$ .

## An Isolating block.

Example :  $\dot{x} = ax$  ( $a < 0$ ),  $\dot{y} = by$  ( $b > 0$ )*vertical lines* : entrance $N^+ :=$  the exit $=$  *horizontal lines* + *vertices* $(N^+$  is closed.) $\Rightarrow N$  is an **isolating block**.The pair  $(N, N^+)$  is called an "index pair".

## Definition

- $S$  : *isolated invariant set*
- $(N_1, N_2)$  : *index pair of  $S$  in an isol. nbh.  $N$  where " $\pi$  is asymptotically compact"*

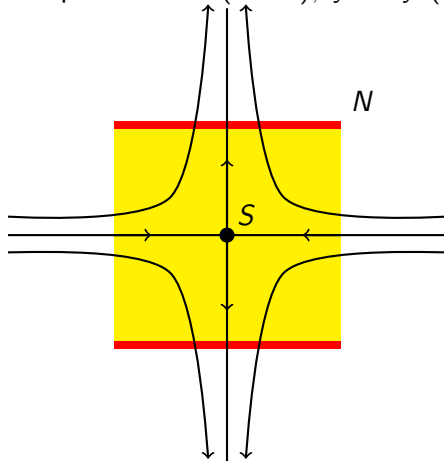
⇒ **the Conley-type index** is the homology group

$$CH_*(S, \pi) = H_*(N_1/N_2, [N_2]).$$

- $\pi$  : asymptotically compact on  $N \Rightarrow S = \text{Inv}(N)$  : compact
- $CH_*(S, \pi)$  does not depend on the choice of index pairs of  $S$ .
- $CH_*(S, \pi) \not\cong 0 \Rightarrow S \neq \emptyset$ .

## The Conley-type index

Example :  $\dot{x} = ax$  ( $a < 0$ ),  $\dot{y} = by$  ( $b > 0$ )



$(N, N^+)$  : index pair

$$CH_n(S) \cong \begin{cases} R & n = 1 \\ 0 & \text{otherwise} \end{cases}$$



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We consider the strongly damped wave equation on a Hilbert space  $H$ :

$$u_{tt} = \alpha \Delta u_t + \Delta u + f(u) \text{ in } \Omega \quad (1)$$

with

$$u = 0 \text{ on } \partial\Omega$$

$\alpha > 0$  : a constant.

$\Omega \subset \mathbb{R}^n$  : a bounded domain

$f : U(\subset X, \text{ open}) \rightarrow Y : \sup_{|u| \rightarrow \infty} (f(u)/u) \leq 0$

$X \subset Y \subset H$  : Hilbert spaces

The strongly damped wave equation (1) can be rewritten by

$$\dot{V} = DU + G(U), \quad (2)$$

$$V = (u, v)^T, \quad D = \begin{pmatrix} 0 & I \\ -A & -\alpha A \end{pmatrix}, \quad G(U) = (0, f(u))^T.$$

$A = -\Delta$ .  $D(A)$  includes the boundary condition.

Now we transform  $(u, v)$  into  $(\phi, \psi)$ , where

$$\phi = u, \psi = \gamma u + \delta v.$$

Then, (2) is equivalent to

$$\dot{U} = K(U) := CU + F(U), \quad (3)$$

$$U = (\phi, \psi)^T, C = \begin{pmatrix} -\gamma\delta^{-1} & \delta^{-1} \\ -\gamma^2\delta^{-1} + (\alpha\gamma - \delta)A & \gamma\delta^{-1} - \alpha A \end{pmatrix},$$

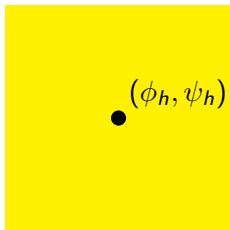
$$F(U) = (0, \delta f(\phi))^T.$$

We go back to the following system in  $X = H_0^1(\Omega) \times H_0^1(\Omega)$  :

$$\begin{cases} \dot{p} = P_h K(p, q) \\ \dot{q} = (I - P_h) K(p, q) \end{cases}, \quad K(p, q) = K(U) = CU + F(U).$$

- $X_h$  : **Finite Element Subspace**,  $P_h$ : ortho. proj. onto  $X_h$
- $p = P_h U$ ,  $q = (I - P_h)U$
- $N := N_1 \times N_2 \subset X$
- $N_1 = \prod_{i=1}^{\dim X_h} [a_i^-, a_i^+] \subset X_h$ ,  $(\phi_h, \psi_h)$  (approximation)  $\in N_1$
- $N_2 = \{(q_1, q_2) \in (I - P_h)X \mid$   
 $(\|q_1\|_{H_0^1}^2 + C_0 \|q_2\|_{L^2}^2)^{1/2} \leq M\}$
- $\pi$  : semiflow for (3)
- $\pi_q$  : semiflow for  $\dot{p} = P_h F(p, q)$ ,  $q \in N_2$

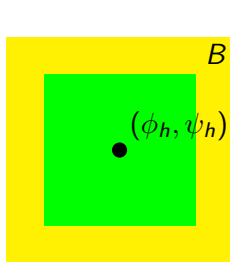
## Scenario of verification

 $N$ 

$(\phi_h, \psi_h)$  : approx. stationary sol.

$N$  : given set

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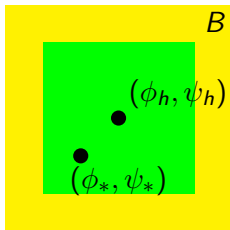


$(\phi_h, \psi_h)$  : approx. stationary sol.

$N$  : given set

$B$  : candidate of isolating block  
with  $CH_*(Inv(B)) \neq 0$

## Scenario of verification



$N$

$(\phi_h, \psi_h)$  : approx. stationary sol.

$N$  : given set

$B$  : candidate of isolating block  
with  $CH_*(Inv(B)) \neq 0$

If  $B \subset N \Rightarrow \exists(\phi_*, \psi_*) \in int(B)$  :  
stationary sol. of original PDE



Next step : Lifting

- Restriction of the dynamics for the tail  
⇒ We know the infinite dimensional dynamics from the finite dimensional one *AUTOMATICALLY*.
- As for the Conley index, **the exit** is essential.

## Fact

*R.Mañé : "The dimension of the global attractor (for (1)) is finite."*

## Next step : Lifting

- Restriction of the dynamics for the tail  
 ⇒ We know the infinite dimensional dynamics from the finite dimensional one *AUTOMATICALLY*.
- As for the Conley index, **the exit** is essential.  
 ⇒ **NO EXIT in the tail**.

## Entrance Condition

*We assume that*

$$\frac{d}{dt}(\|q_1\|_{H_0^1}^2 + C_0\|q_2\|_{L^2}^2) < 0 \quad (4)$$

for all  $(\phi, \psi) \in N$  with  $(\|q_1\|_{H_0^1}^2 + C_0\|q_2\|_{L^2}^2)^{1/2} =$   
 $(\|(I - P_h)\phi\|_{H_0^1}^2 + C_0\|(I - P_h)\psi\|_{L^2}^2)^{1/2} = M.$

## Entrance Condition

We assume that

$$\frac{d}{dt}(\|q_1\|_{H_0^1}^2 + C_0\|q_2\|_{L^2}^2) < 0 \quad (5)$$

for all  $(\phi, \psi) \in N$  with  $(\|q_1\|_{H_0^1}^2 + C_0\|q_2\|_{L^2}^2)^{1/2} =$   
 $(\|(I - P_h)\phi\|_{H_0^1}^2 + C_0\|(I - P_h)\psi\|_{L^2}^2)^{1/2} = M.$

## Theorem (M.)

Let  $N = N_1 \times N_2$  be a strongly  $\pi$ -admissible closed set in  $X$ . We assume that (5) holds.

- 1 If  $N_1$  is an isol. block for  $\pi_q$  for all  $q \in N_2$ , then so is  $N$  for  $\pi$ .
- 2  $CH_*(\text{Inv}(N), \pi) = CH_*(\text{Inv}(N_1), \pi_q)$  for all  $q$ .

We set

$$\gamma := 2/\alpha, \quad \delta := 2, \quad C_0 = C_0(\alpha, h) \approx 1$$

so that we translate (5) into “an inequality”.

## Proposition (M.)

*Condition (5) holds if the following inequality is satisfied:*

$$\gamma_*(h)(2\lambda \sup_{u \in N} \|f(u)\|_{L^2} + F_N) < M, \quad (6)$$

where

$$\begin{aligned} \gamma_*(h) &= 2(0.5 - 4C_0(Ch)^2\alpha^{-2})C_0Ch, \\ F_N &= F_N(\gamma, \delta, \phi, \psi) : \text{computable}, \end{aligned}$$

$C$  : a constant depending on the choice of the basis on  $X_h$ .  $h$  : the mesh size.

## Main theorem

### Theorem (M.)

Let

$$\begin{cases} N_1 := \{(\phi_h, \psi_h)\} + \prod_{k=1}^{\dim X_h} [a_k^-, a_k^+], \\ N_2 := [M] = \{q \in (I - P_h)X \mid \|q_1\|_{H_0^1}^2 + C_0 \|q_2\|_{L^2}^2 \leq M^2\}. \end{cases}$$

If there exists an isolating block  $B$  in  $N_1$  such that

- $CH_n(\text{Inv}(B), \pi_q) \cong \begin{cases} R & n = \ell \text{ for some } \ell \\ 0 & \text{otherwise} \end{cases}, \forall q \in N_2$
- $\gamma_*(h)(2\lambda \sup_{u \in N} \|f(u)\|_{L^2} + F_N) < M$

hold, then there exists an equilibrium of (3) in  $N := N_1 \times N_2$ .

Next step : *Construction of ISOLATING BLOCK*

$$\dot{U} = K(U) \Rightarrow \begin{cases} \dot{p} = P_h K(U) & \Leftarrow \text{Conley-type index} \\ \dot{q} = (I - P_h)K(U) & \Leftarrow \text{Entrance condition} \end{cases}$$

$\Rightarrow$  Consider the dynamics of the lower term **around**  $u_h$ .

$\Rightarrow$  Transform the original system to the perturbed diagonalized system.

Original System  $\Rightarrow$  Perturbed Diagonalized System  
around  $(\phi_h, \psi_h)$ . A solution  $U = (\phi, \psi)^T$  satisfies

$$(\phi_t, v) = (-\gamma\delta^{-1}\phi + \delta^{-1}\psi, v)$$

$$(\psi_t, w) = (-\gamma^2\delta^{-1}\phi + \gamma\delta^{-1}\psi, w) - ((\alpha\gamma - \delta)\nabla\phi + \alpha\nabla\psi, \nabla w) \\ + \delta(f(\phi), w), \text{ for all } (v, w) \in X.$$

$\Rightarrow$  Finite dimensional projection:

$$(\phi_t, v_h) = (-\gamma\delta^{-1}\phi + \delta^{-1}\psi, v_h)$$

$$(\psi_t, w_h) = (-\gamma^2\delta^{-1}\phi + \gamma\delta^{-1}\psi, w_h) - ((\alpha\gamma - \delta)\nabla\phi + \alpha\nabla\psi, \nabla w_h) \\ + \delta(f(\phi), w_h), \text{ for all } (v_h, w_h) \in X_h.$$



If we set  $(\tilde{\phi}, \tilde{\psi}) := (\phi - \phi_h, \psi - \psi_h)$ , it must satisfy

$$\begin{aligned} (\nabla(\tilde{\phi}_h)_t, \nabla v_h) &= (-\gamma\delta^{-1}\nabla\tilde{\phi} + \delta^{-1}\nabla\tilde{\psi}, \nabla v_h) \\ ((\tilde{\psi}_h)_t, w_h) &\in (-\gamma^2\delta^{-1}\tilde{\phi} + \gamma\delta^{-1}\tilde{\psi}, w_h) - ((\alpha\gamma - \delta)\nabla\tilde{\phi} + \alpha\nabla\tilde{\psi}, \nabla w_h) \\ &\quad + \delta(f(\phi), w_h), \text{ for all } (v_h, w_h) \in X_h. \end{aligned}$$

$\Rightarrow$  (“diagonalization”)

$$\begin{aligned} \dot{Y}_h &\in \Lambda Y_h + Q\Phi^{-1}(\epsilon_1(w), \dots, \epsilon_{\dim X_h}(w))^T. \\ (\dot{y}_i &\in \lambda_i y_i + \tilde{\epsilon}_i(\tilde{y}), \quad i = 1, \dots, \dim X_h) \end{aligned}$$

- $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{\dim X_h})$ .
- $Q, \Phi$  : nonsingular linear transformations.
- $\tilde{\epsilon}_i(\tilde{y}) = o(|y|^2)$  (as  $|y| \rightarrow 0$ ) : the  $i$ -th bound of error terms.

Set  $\tilde{\epsilon}_i(\tilde{y}) := [\delta_i^-, \delta_i^+]$ .  $\dot{y}_i$  must satisfy the following relation:

$$\lambda_i \left( y_i + \frac{\delta_i^-}{\lambda_i} \right) < \dot{y}_i < \lambda_i \left( y_i + \frac{\delta_i^+}{\lambda_i} \right).$$

$\Rightarrow$  the candidate of our isolating neighborhood is as follows:

$$\begin{cases} \tilde{W}_i^{(k)} := \left[ -\frac{\delta_i^+}{\lambda_i}, -\frac{\delta_i^-}{\lambda_i} \right], & \text{if } \lambda_i > 0, \\ \tilde{W}_i^{(k)} := \left[ -\frac{\delta_i^-}{\lambda_i}, -\frac{\delta_i^+}{\lambda_i} \right], & \text{if } \lambda_i < 0. \end{cases} \quad (7)$$

### Remark

*Our isolating block is  $\left( \{(\phi_h, \psi_h)\} + \prod_{i=1}^{\dim X_h} Q^{-1} \tilde{W}_i^{(k)} \right) \times [M]$ .  
 To compute the CR-index, we just count the number of  $i \in \{1, \dots, \dim X_h\}$  such that  $\lambda_i > 0$ .*

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Hyperbolic equilibrium.

We consider an evolutionary equation

$$\dot{u} = -Au + f(u) \quad (8)$$

on a (separable) Hilbert space  $X$ .

- $A$  : positive-definite self-adjoint (generally, sectorial)
- $A^{-1}$  : compact
- $f : D(A^\alpha) \rightarrow X : C^1$ .
- $\forall B \subset D(A^\alpha) : \text{bounded} \Rightarrow f(B) \subset X : \text{bounded}$ .

## Definition

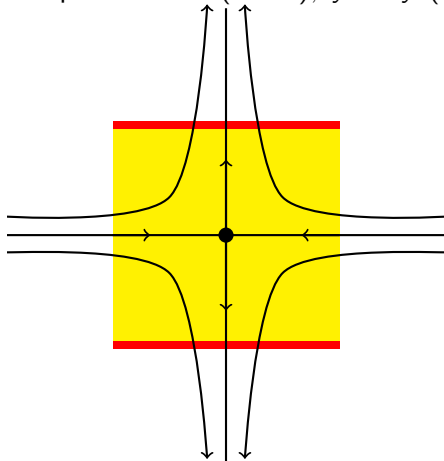
We say an equilibrium  $\bar{u}$  for (8) is *hyperbolic* if, for the linearization  $L := -A + df(\bar{u})$  at  $\bar{u}$ ,

$$\sigma(L) \cap \sqrt{-1}\mathbb{R} = \emptyset$$

holds, where  $\sigma(L)$  is the spectrum of  $L$ .

## Hyperbolic equilibrium

Example :  $\dot{x} = ax$  ( $a < 0$ ),  $\dot{y} = by$  ( $b > 0$ )



$$(\sigma(L) = )\{a, b\} \cap \sqrt{-1}\mathbb{R} = \emptyset.$$

$\{0\}(= S)$  is hyp. equilibrium  
with  $\dim W^u(S) = 1$ .

## Preceding results for verifying equilibria of PDEs ...

- Existence
- Stability (e.g. Conley index)
- Uniqueness

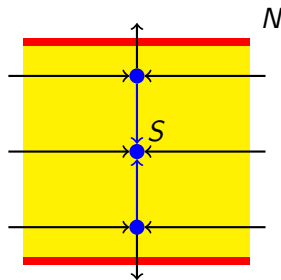
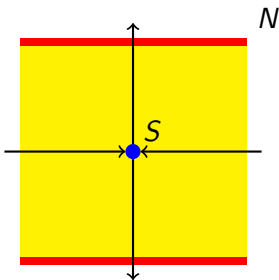
of equilibria in a subset  $N$  of a Banach space.

(Zgliczyński, Gameiro, Lessard, Mischaikow, ...)

(Nakao et. al., Oishi et. al.)

## “Existence of equilibria”

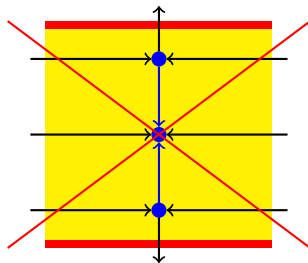
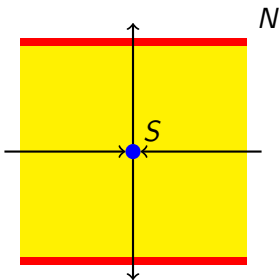
(Schauder’s fixed point theorem, Conley index, Brouwer’s degree)



At least one equilibrium in  $N$ . Which is the true situation ?



## “Existence + Local uniqueness” (Contraction mapping principle)

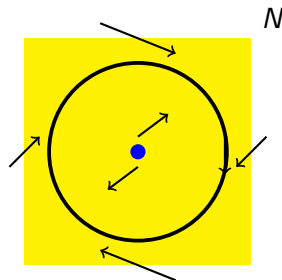
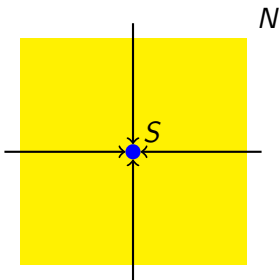


Equilibrium is unique in  $N$ . NOT the case of the right figure.

As an invariant set of dynamical systems . . .

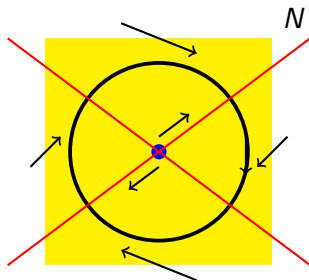
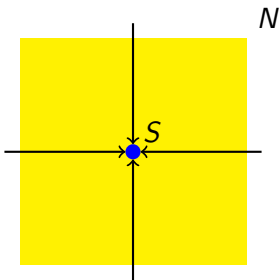
- Gradient dynamics : Inv. sets = equilibria + connecting orbits
- General dynamics : + “recurrent” inv. sets (e.g. periodic orbits, chaotic inv. sets)

## Gradient dynamics vs. General dynamics



Even if an equilibrium  $\bar{u}$  is unique in  $N$ ,  $\{\bar{u}\} \neq \text{Inv}(N)$  in general.  
 $\Rightarrow$  Stability of  $\{\bar{u}\} \neq$  Stability of  $\text{Inv}(N)$

If dynamics in  $N$  is gradient-like ...



All recurrent invariant sets (e.g. periodic orbits) are excluded.  
 $\Rightarrow$  Precise structure of  $Inv(N)$

## Our requirements ...

- Local uniqueness of equilibrium
- Hyperbolicity of equilibrium
- Construction of **Lyapunov function** on  $N$   
(so that dynamics is gradient-like in  $N$ )

## Hyperbolicity verification theorem.

Two types of verification theorems

depending on rigorous numerical verification method.

- 1 Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) := d_i u_i + N_i(u), \quad i = 1, 2, \dots .$$

- 2 Other form.  
(In case that  $u$  is not always expanded by eigenfunctions)

## The Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi] \quad (9)$$

$$u(t, -x) = -u(t, x) \text{ with per. B.C.} \quad (10)$$

Using the Fourier basis  $\{\sin(k\pi x)\}_{k \geq 1}$ ,

$u(t, x) = -2 \sum_{k \in \mathbb{N}} u_k \sin(k\pi x)$  and (9)+(10) is rewritten by

$$\dot{u}_k = k^2(1 - \nu k^2)u_k - k \sum_{n=1}^{k-1} u_n u_{k-n} + 2k \sum_{n=1}^{\infty} u_n u_{n+k},$$

which forms

$$\dot{u}_k = F_k(u) = d_k u_k + N_k(u), \quad k = 1, 2, \dots$$

## Rigorous verification of equilibria

ZM-theory (Zgliczynski and Mischaikow, Found. Comp. Math. (2001) 255–288.)

Rigorous verification of equilibria of equations which form

$$\dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \dots . \quad (11)$$

⇒ Find a set  $V$  (in a Hilbert space  $X$ ) which forms

$$V = \prod_{k=1}^n [w_k^-, w_k^+] \times \prod_{k>n} \left[ -\frac{C}{k^s}, \frac{C}{k^s} \right] \quad (w_k^\pm \in \mathbb{R}, C > 0, s \in \mathbb{N}) \quad (12)$$

containing an equilibrium  $u^*$  of (11).

(*Tool* : Conley-type index or mapping degree)



## Hyperbolicity verification theorem.

We consider an evolutionary equation on  $X$ :

$$\dot{u} = F(u) \Leftrightarrow \dot{u}_i = F_i(u) = d_i u_i + N_i(u), \quad i = 1, 2, \dots \quad (13)$$

- $d_i < 0$  for all sufficiently large  $i$ .
- $u = (u_1, u_2, \dots) \in X$ ,  $V \subset X$  which forms (12).
- $\partial F_i / \partial u_j \in C(V, \mathbb{R})$ ,  
 $\sum_{j \geq 1} \max_{u \in V} |(\partial F_i / \partial u_j)(u)| \sup_{x, y \in V} |x_j - y_j| < \infty$ .

### Theorem (Zgliczyński)

If, for all  $i \in \mathbb{N}$ ,

$$\inf_{u \in V} \left| \frac{\partial F_i}{\partial x_i}(u) \right| > \sum_{j \neq i} \sup_{u \in V} \left| \frac{\partial F_i}{\partial x_j}(u) \right|$$

holds, then  $F(x) = 0$  has at most one solution in  $V$ .

## Assumption

$$\sigma(DF(u)) \cap \sqrt{-1}\mathbb{R} = \emptyset \quad (14)$$

*holds for all  $u \in V$ .*

## Theorem (M.)

Let  $u^*$  be an equilibrium of (13) in  $V$ . If (14) holds and

$$\inf_{i \in \mathbb{N}} [ |d_i| - \sum_{j \geq 1} \sup_{u \in V} \left| \frac{\partial N_i}{\partial x_j}(u) \right| ] = \delta_1 > 0, \quad (15)$$

$$\inf_{i \in \mathbb{N}} [ |d_i| - \sum_{j \geq 1} \sup_{u \in V} \left| \frac{\partial N_j}{\partial x_i}(u) \right| ] = \delta_2 > 0 \quad (16)$$

$$m := \text{the number of } d_i \text{ with positive real part} \quad (17)$$

hold, then  $\text{Inv}(V) = \{u^*\}$ .  $u^*$  is hyperbolic (for (13)) with  $\dim W^u(\{u^*\}) = m$ .

## Outline of proof

*Construct a Lyapunov function of the form*

$$L(u) := - \sum_{i \geq 1} \text{sign}(d_i) \cdot (u_i - u_i^*)^2,$$

*where  $u^*$  is the unique equilibrium of  $\dot{u} = F(u)$  in  $V$ .*

- Define  $G(u) := \frac{dL}{dt}(u)$ .

## Outline of proof

$$\frac{1}{2} \frac{\partial G}{\partial u_i} = -\text{sign}(d_i) \left\{ 2d_i(u_i - u_i^*) + (1 + \delta_{ij}) \sum_{j \geq 1} \frac{\partial N_i(c_{ij})}{\partial u_j} (u_j - u_j^*) \right\}$$

$$\frac{1}{2} \frac{\partial^2 G}{\partial u_i \partial u_j} = -\text{sign}(d_i) \left\{ 2d_i \delta_{ij} + \left( \frac{\partial N_i(c_{ij})}{\partial u_j} + \frac{\partial N_j(c_{ji})}{\partial u_i} \right) \right\}$$

By (15) and (16),

- $u$  is the critical point of  $G \Leftrightarrow u = u^*$ .
- $G$  is strictly negative-definite.  $G(u) = 0$  iff  $u = u^*$  (in  $V$ ).
- $G \leq 0$ . Thus  $L$  is a Lyapunov function in  $V$ .

In case of FEM...

Cannot expand  $u$  by *eigenfunctions*.

$$u = \sum_{i=1}^n u_i \varphi_i + \varphi_{\perp}, \quad \varphi_{\perp} \in S_h^{\perp}$$

- Behavior of Lyapunov function  
 $\Rightarrow$  perturbed diagonal system + *entrance condition* (M.)
- No information about spectrum in the tail term ... ?  
 $\Rightarrow$  *the Conley-type index* (M.)

- 1 Preliminaries
  - “Rigorous” numerics
  - The Conley-type index
- 2 Validated computation
  - Lifting
  - The tail term
  - The lower term
- 3 Local dynamics around equilibria
  - Hyperbolic equilibrium
  - Preceding results and Our requirements
  - Hyperbolicity verification theorem
- 4 Application
  - Rigorous numerical verification methods
  - Examples

## Rigorous numerical verification of hyperbolic equilibria

### Existence of equilibria

ZM-theory

(or “FEM + the Conley-type index” (M.)).

### Lyapunov function

Main theorem which is shown before.

### Eigenvalue exclusion (= hyperbolicity)

Nakao's theory.

Ref. : M.T. Nakao, K. Hashimoto and Y. Watanabe,  
*Computing*, **75**(2005), 1–14.



Ex. : the Kuramoto-Sivashinsky equation

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \quad \text{for } t \geq 0, x \in [-\pi, \pi],$$

$$u(t, x) = -u(t, -x), u(t, x + 2\pi) = u(t, x).$$

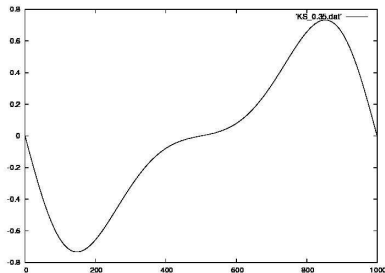
Remark

$$u^* \in \{u_{32}\} + \prod_{k=1}^{32} [-w_k, w_k] \times \prod_{k>32} \left[ -\frac{C}{k^s}, \frac{C}{k^s} \right]$$

Equilibrium  $u_*$  of  $u_t = -0.35u_{xxxx} - u_{xx} + 2uu_x$  in  $[-\pi, \pi]$

$$C = 2040.74723, \quad s = 6, \quad \sup_{1 \leq k \leq 32} w_k \leq 3.59468318 \times 10^{-5}$$

$$\dim W^u(u^*) = 0.$$



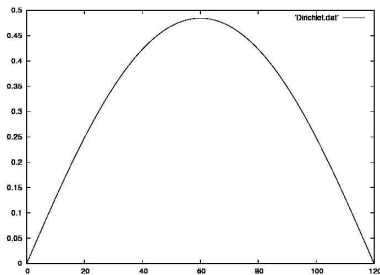
Ex. : Strongly damped wave equation (0-Dirichlet)

$$u_{tt} = u_{txx} + u_{xx} + \lambda(u - u^3) \text{ in } I = (0, 1)$$
$$u(t, 0) = u(t, 1) = 0.$$

Equilibrium  $u_*$  of  $u_{tt} = u_{txx} + u_{xx} + 12(u - u^3)$  in  $I = (0, 1)$

$$\sup_{1 \leq k \leq 60} w_k \leq 4.11387465 \times 10^{-4}, \quad M = 4.0619661 \times 10^{-4}$$

$$\dim W^u(u^*) = 0.$$



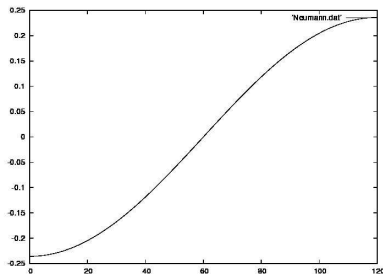
Ex. : Strongly damped wave equation (0-Neumann)

$$u_{tt} = u_{txx} + u_{xx} + \lambda(u - u^3) \text{ in } I = (0, 1)$$
$$u_x(t, 0) = u_x(t, 1) = 0.$$

Equilibrium  $u_*$  of  $u_{tt} = u_{txx} + u_{xx} + 10.3(u - u^3)$  in  $I = (0, 1)$

$$\sup_{1 \leq k \leq 60} w_k \leq 2.2024165 \times 10^{-4}, \quad M = 2.0053294 \times 10^{-4}$$

$$\dim W^u(u^*) = 1.$$



Future works:

- **Connecting orbits between stationary solutions of PDEs.**

- Hyperbolicity + covering relation, cone condition  
⇒ description of stable and unstable manifolds
- Known numerical verification methods  
⇒ ODE-like methods to PDEs.

- **Hyperbolic periodic orbits of PDEs.**

- Rigorous computation of Poincaré maps (Zgliczyński)
- Topological tools (covering relation + discrete Conley index)  
⇒ computation of  $\dim W^u(\gamma)$  (M.)

⇒ **GLOBAL DYNAMICS**

⇒ **Further applications (Bifurcation problems)**