

# Rigorous verification of bifurcations of differential equations via the Conley index theory

Kaname Matsue \*

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## Abstract

We propose a new approach for capturing bifurcations of (semi)flows by using a topological tool, the Conley index. We can apply this concept to capture bifurcations with rigorous numerics. As an example, we consider the dynamics generated by the Swift-Hohenberg PDE and show that a pitchfork-like bifurcation occurs in a certain region.

**Keywords:** Conley index, saddle-node and pitchfork bifurcation, rigorous numerics.

## 1 Introduction

The goal of this paper is to give a new topological method for rigorously capturing bifurcations.

Nonlinear phenomena in various fields such as physics and engineering are often described by differential equations, and the study of their solution structure is essential for a good understanding of the nonlinear phenomena. However, due to their nonlinearity, it is often the case that conventional mathematical methods are limited for analyzing properties of the solutions of the differential equations, especially, their global structure in the phase space or its changes when parameters are varied. In such cases, numerical methods are widely used for understanding the behaviors of the solutions, but one has to be careful about the correctness of the conclusions derived by numerical methods because of the presence of errors that numerical methods inevitably contain.

Recently, the so-called “self-validating numerical methods” for differential equations using the interval arithmetics have been developed in order to guarantee the mathematical rigorousness of numerical results. Moreover, such a rigorous computation can be combined with some topological method to study not only the existence of some specific solutions of a differential equation but also its dynamics. The Conley index theory is one of the possible topological methods for such a purpose. The Conley index is defined for a specific type of invariant sets called “isolated invariant sets”. This is an invariant set contained in the interior of its compact neighborhood called an isolating neighborhood, hence it is isolated from its outside. Roughly, the Conley index of an isolated invariant set  $S$  is the relative homology of an isolating neighborhood of  $S$  and its exit set. See §2 for a precise definition.

The combination of the rigorous computation and the Conley index theory has been successfully applied to the existence of stationary solutions of some partial differential equations such as the Kuramoto-Sivashinsky equation ([22]) and the Swift-Hohenberg equation ([4]). Moreover, this method was used to follow branches of the stationary solutions by a (large) variation of parameters as long as no bifurcation occurs ([13]). See also [2], [5] and [10].

At a bifurcation point, the branch-following method mentioned above fails due to the occurrence of the zero eigenvalue, and hence one needs to treat such bifurcation points differently when one tries to numerically follow solutions through bifurcation. Indeed, several methods have been developed for treating bifurcations points numerically, see e.g. [6].

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\*Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan (k-matsue@math.kyoto-u.ac.jp).

A new topological-numerical approach for studying global dynamics and changes in parameters has recently been proposed in [1], where the dynamics of multi-parameter systems is expressed in terms of a directed graph with a certain algebraic information, called *the Conley-Morse graph*. The directed graph represents a Morse decomposition of the dynamics and an associated algebraic information, the Conley index of each Morse set. One may thus discuss changes of dynamics at different parameter values by comparing the corresponding Conley-Morse graphs. This requires a new idea for viewing changes of dynamics, based on topological and discrete data that represent dynamics. For instance, if one observes that, in a region of the phase space which remain isolated over an interval of a parameter, there exists a pair of invariant sets at a parameter value that changes to an empty invariant set at another parameter value, one may suspect that a certain bifurcation occurs, and moreover, it may look like a saddle-node bifurcation. Such a change of dynamics may easily be observed by comparing Conley-Morse graphs as discussed in [1], and hence it may be useful if there is a mathematical theory that tells what kind of “bifurcations” occurs in the variation of the parameter.

This paper is an attempt toward such a direction. We shall introduce a coarser notion of bifurcations of stationary solutions, which we call the *C-saddle-node* and *C-pitchfork* bifurcation, which are weaker notions of the saddle-node and pitchfork bifurcations in the usual sense. For a given parameterized semiflow, we define a topological framework, in terms of an isolating neighborhood of the parameterized semiflow, in which such weaker notions of bifurcations are forced to occur. This framework is called the isolating neighborhood of *C-saddle-node type* or of *C-pitchfork type* and these notions describe *the change of gradient-like structure of dynamics*. We show that a self-validating computational method combined with the Conley index theory can be used to rigorously detect such C-type isolating neighborhoods.

Our main results may be summarized as follows:

**(I)** We give precise definitions of two C-type isolating neighborhoods and prove that the invariant set in these isolating neighborhoods changes its structure in a way similar to the saddle-node or the pitchfork bifurcations for stationary solutions. See §3.

**(II)** We apply the above topological framework to the Swift-Hohenberg equation and verify that the C-pitchfork bifurcation does occur from stationary solutions. Here we would like to consider bifurcations as the change of gradient-like structure of dynamics. Hence we do not use well-known analytic methods for verifying bifurcations, such as the Lyapunov-Schmidt reduction, in our present verification. See §5 for details.

This paper is organized as follows.

In §2, we briefly recall the definition of the Conley index and its related notions. In §3, we introduce the C-type isolating neighborhoods and prove their properties. In §4, we review rigorous numerical methods which we use in our application, §5, briefly. In §5, we provide several rigorous numerical results of bifurcations for Swift-Hohenberg PDE. In the end of this paper, we list tables of numerical results.

## 2 The Conley index

In this section, let  $\Gamma$  be a metric space and  $\varphi : \mathbb{R}_{\geq 0} \times \Gamma \rightarrow \Gamma$  be a global semiflow on  $\Gamma$  ( $x \cdot t := \varphi(t, x)$ ).

### 2.1 Isolated invariant sets and the Conley index

Recall that a *solution through*  $x \in \Gamma$  is a continuous map

$$\sigma_x : (a, \infty) \rightarrow \Gamma$$

where  $a \in [-\infty, 0)$  such that

- $\sigma_x(0) = x$ ,
- for  $t \in (a, \omega_x)$  and  $s > 0$ , it follows  $\sigma_x(t) \cdot s = \sigma_x(s + t)$ .

We define the invariant part of a subset  $N \subset \Gamma$  as

$$\text{Inv}(N) = \text{Inv}(N, \varphi) := \{x \in N \mid \text{there is a full solution } \sigma_x \text{ through } x \text{ such that } \sigma_x((-\infty, \infty)) \subset N\}.$$

A set  $S \subseteq \Gamma$  is *invariant* if  $\text{Inv}(S) = S$  holds. If  $S \subset \Gamma$  is invariant, then its closure  $\overline{S}$  is also invariant. See Salamon [17] for details.

**Definition 2.1.** A locally compact subset  $X \subset \Gamma$  is said to be a *local flow* if for every  $x \in X$  there exists a neighborhood  $U$  of  $x$  in  $\Gamma$  and an  $\epsilon > 0$  such that  $(X \cap U) \cdot [0, \epsilon) \subset X$ .

Throughout this section, let  $X \subset \Gamma$  be a local flow.

**Definition 2.2.** A compact subset  $N \subset X$  is an *isolating neighborhood in  $X$*  if

$$\text{Inv}(N) \subset \text{int}_X(N),$$

where  $\text{int}_X(N)$  represents the interior of  $N$  in the relative topology of  $X$ . Moreover,  $N$  is called an *isolating block in  $X$* , if, for each  $x \in \partial N$ , there exists  $\epsilon_x > 0$  that satisfies

$$\sigma_x(0, \epsilon_x) \cap N = \emptyset \text{ or } \sigma_x(-\epsilon_x, 0) \cap N = \emptyset$$

for every solutions  $\sigma_x$  through  $x$ .

**Definition 2.3.** A set  $S \subset X$  is an *isolated invariant set* if there exists an isolating neighborhood  $N$  in  $X$  such that  $\text{Inv}(N) = S$ .

The Conley index of isolated invariant sets is defined by a compact pair with special property.

**Definition 2.4.** A compact pair  $(N, L)$  is called an *index pair in  $X$*  if the following conditions are satisfied:

(IP1)  $\overline{(N \setminus L)}$  is an isolating neighborhood in  $X$ .

(IP2)  $L$  is positively invariant in  $N$ . That is, if  $x \in L$  and  $x \cdot [0, t] \subseteq N$ , then  $x \cdot [0, t] \subseteq L$ .

(IP3) If  $x \in N$  and  $x \cdot [0, T] \not\subseteq N$  for some  $T > 0$ , then there exists  $T_0 \in [0, T)$  such that  $x \cdot [0, T_0] \subseteq N$  and  $x \cdot T_0 \in L$ .

**Remark 2.5.** 1. If  $S$  is an isolated invariant set with  $S = \text{Inv}(\overline{N \setminus L})$ ,  $(N, L)$  is called an index pair of  $S$  in  $X$ .

2. By (IP3),  $L$  is often called an exit set.

3. If  $N$  is an isolating block in  $X$  and  $L$  is the exit set,  $(N, L)$  is an index pair in  $X$ .

**Definition 2.6.** Let  $S$  be an isolated invariant set and  $(N, L)$  be an index pair of  $S$  in  $X$ . Then the *Conley index* of  $S$  is given by the homology

$$CH_*(S) = CH_*(S, \varphi) := H_*(N, L; \mathbb{Z}_2).$$

(see [3], [14] and [18].) For our purpose it is easier to work with the coefficient field  $\mathbb{Z}_2$ .

**Remark 2.7.** The definition of isolating neighborhoods, isolated invariant sets and the Conley index are originally considered in case of *flows*. Rybakowski [16] extended the index theory for semiflows. Especially, for many parabolic PDEs, the above definitions are valid if an isolating neighborhood  $N$  is compact. Such an isolating neighborhood is called *strongly admissible*. See [16] for more details. Throughout this paper, “isolating neighborhood” will always mean a strongly admissible isolating neighborhood.

## 2.2 Attractor-repeller pairs, Morse decompositions

Here we present a decomposition of a compact invariant set. We remark that all definitions in this subsection are valid for  $\Gamma$ , a metric space, or, more generally, a Hausdorff topological space. See [17].

An  $\omega$ -*limit set*  $\omega(Y)$  of a subset  $Y$  ([12] p.35) is defined as

$$\omega(Y) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} Y \cdot t} = \text{Inv}(\overline{Y \cdot [0, \infty)}).$$

An  $\alpha$ -limit set  $\alpha(Y)$  of a subset  $Y$  ([12] p.35) is defined by

$$\alpha(Y) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} H(t, Y)} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \left( \bigcup_{y \in Y} H(t, y) \right)},$$

where

$$H(t, x) := \{y \in \Gamma \mid \text{there is a negative orbit through } x \text{ defined by } \sigma : (-\infty, 0] \rightarrow \Gamma \text{ with } \sigma(0) = x \text{ and } \sigma(-t) = y\}.$$

**Definition 2.8.** Let  $S$  be a compact invariant subset in  $\Gamma$ . Then  $A \subset S$  is an *attractor* in  $S$  if there exists an open neighborhood  $U$  of  $A$  in  $S$  such that

$$A = \omega(U).$$

If  $A$  is an attractor in  $S$ , the *dual repeller* of  $A$  in  $S$  is defined by

$$A^* := \{x \in S \mid \omega(x) \cap A = \emptyset\}.$$

Similarly, a subset  $R \subset S$  is called an *repeller* in  $S$  if there exists an open neighborhood  $V$  of  $R$  such that

$$R = \alpha(V).$$

If  $R$  is an attractor in  $S$ , the *dual attractor* of  $R$  in  $S$  is defined by

$$R^* := \{x \in S \mid \alpha(x) \cap R = \emptyset\}.$$

Let  $S_1, S_2$  be invariant subsets of  $S$ . Then we define the set of *connecting orbits* from  $S_2$  to  $S_1$  as

$$C(S_1, S_2; S) := \{x \in S \mid \alpha(x) \subset S_2, \omega(x) \subset S_1\}.$$

If  $S$  is a compact invariant set and  $A$  is an attractor in  $S$ , then  $S$  is decomposed into the union  $S = A \cup A^* \cup C(A, A^*; S)$ .

**Definition 2.9.** Let  $S$  be a compact invariant subset of  $\Gamma$  and  $A$  be an attractor in  $S$ . Then a pair  $(A, A^*)$  is called an *attractor-repeller pair* (*A-R pair*) in  $S$ .

We present the definition of a Morse decomposition, which is a gradient-like decomposition of a compact invariant set. Moreover, we present the definition of an interval which is a subset of a partially ordered set. A Morse decomposition with a partially ordered set  $P$  and an interval in  $P$  are closely related. See [14] and [17].

**Definition 2.10.** Let  $S$  be a compact invariant set (It is not necessary isolated.) and  $P$  be a finite set. Then a collection

$$\{M(p) \mid p \in P\}$$

of disjoint compact invariant sets is called a *Morse decomposition of  $S$*  if there exists a strict partial order  $>$  such that, for all

$$x \in S \setminus \bigcup_{p \in P} M(p),$$

there exist  $p, q \in P$  with  $p > q$  which satisfies

$$\omega(x) \subset M(q) \text{ and } \alpha(x) \subset M(p).$$

**Remark 2.11.** In this definition, it is not assumed that there is a unique partial order on  $P$ . In general, any ordering on  $P$  satisfying the above property is called *admissible* (for the flow). Now we can identify an ordering  $<$  on the collection  $\{M(p) \mid p \in P\}$  associated to an admissible ordering. Namely, we define  $M(p) > M(q)$  holds if  $p > q$  holds with respect to an admissible ordering  $<$  on  $P$ .

**Remark 2.12.** If  $\{M(p) \mid p \in P\}$  is a Morse decomposition of  $S$ , each  $M(p)$  is called Morse set. Moreover, if  $S$  is isolated then each  $M(p)$  is also isolated. See [17] or [18].

**Definition 2.13.** Let  $(P, <)$  be a partially ordered set. A subset  $I \subset P$  is called an *interval* if  $p < r < q$  with  $p, q \in I$  implies  $r \in I$ . We say  $p, q \in P$  are *adjacent* with respect to  $<$  if either  $\{p, q\}$  or  $\{q, p\}$  is an interval in  $P$ .

## 2.3 Connection matrices

Here we present the connection matrix of a Morse decomposition. This matrix gives an algebraic information of a Morse decomposition and is useful to know the information of connecting orbits between Morse sets.

**Definition 2.14.** Let  $S$  be an isolated invariant set (in a local flow  $X$ ) and the collection of invariant sets  $\{M(p) \mid p \in (P, \succ)\}$  be a Morse decomposition of  $S$  with admissible ordering  $\succ$ . Then an associated *connection matrix* is a linear map

$$\Delta : \bigoplus_{p \in P} CH_*(M(p)) \rightarrow \bigoplus_{p \in P} CH_*(M(p)),$$

where

$$\Delta(p, q) : CH_*(M(q)) \rightarrow CH_*(M(p))$$

is the corresponding  $(p, q)$ -component of  $\Delta$ , such that the following conditions are satisfied:

**(CM1)**  $\Delta$  is upper triangular. in the sense that if  $p \not\succeq q$  then

$$\Delta(p, q)CH_*(M(q)) = 0.$$

**(CM2)**  $\Delta$  is a boundary operator; that is,

$$\Delta(p, q)CH_n(M(q)) \subset CH_{n-1}(M(p))$$

and  $\Delta \circ \Delta = 0$ .

**(CM3)** The Conley indices of the Morse sets and the Conley index of the total isolated invariant set  $S$  satisfy the following relation:

$$\frac{\text{Ker}\Delta}{\text{Im}\Delta} \cong CH_*(S).$$

**Remark 2.15.** The above definition may be considered as an axiomatic definition of the connection matrix. In order to make the definition meaningful, one must show the existence of such a connection matrix, which is given in [8]. For the use of connection matrices, the properties (CM1)-(CM3) are enough for applications.

## 3 C-type isolating neighborhoods

In this section we introduce two new concepts for capturing bifurcations of a semiflow using a topological approach. Let  $X$  be a locally compact metric space (in general, a local flow of the semiflow  $\varphi^\lambda$ ),  $\Lambda \subset \mathbb{R}$  be a compact interval and  $\Phi : \mathbb{R}_{\geq 0} \times X \times \Lambda \rightarrow X \times \Lambda$  be parameterized semiflow on  $X$ :

$$\Phi(t, u, \lambda) := (\varphi^\lambda(t, u), \lambda).$$

Let  $N \subset X \times \Lambda$  and let  $N^\lambda := N \cap (X \times \{\lambda\})$ .

### 3.1 Saddle-node type

In this subsection, we introduce a weaker notion of saddle-node bifurcation in the view of the Conley index theory.

**Definition 3.1.** Let  $N$  be a connected isolating neighborhood for  $\Phi$  such that  $N^\lambda$  is connected for each  $\lambda$ . We say  $N$  is of *C-saddle-node type over*  $[\lambda_-, \lambda_+]$  ( $\subset \Lambda$ ) if the following conditions are satisfied:

**(CSN1)** For each isolated invariant set  $S$  in  $\text{Inv}(N^{\lambda_-})$ ,  $S$  satisfies

$$CH_*(S, \varphi^{\lambda_-}) \cong 0.$$

(CSN2) There exist disjoint isolating neighborhoods  $N_0^{\lambda+}$  and  $N_1^{\lambda+}$  in  $N^{\lambda+}$  such that

$$CH_*(\text{Inv}(N_0^{\lambda+}), \varphi^{\lambda+}) \not\cong 0,$$

$$CH_*(\text{Inv}(N_1^{\lambda+}), \varphi^{\lambda+}) \not\cong 0$$

and  $\{\text{Inv}(N_0^{\lambda+}), \text{Inv}(N_1^{\lambda+})\}$  forms an A-R decomposition of  $\text{Inv}(N^{\lambda+})$ .

We remark that if  $N$  is of C-saddle-node type,  $N$  also satisfies

$$CH_*(\text{Inv}(N^\lambda)) \cong 0$$

for all  $\lambda \in [\lambda_-, \lambda_+]$  by the continuation property.

Now we prove that there exists  $\lambda^* \in (\lambda_-, \lambda_+)$  such that the dynamics undergoes a saddle-node-like bifurcation at  $\lambda^*$  if  $N$  is an isolating neighborhood of C-saddle-node type over  $[\lambda_-, \lambda_+]$ .

Let  $\Lambda_{\text{SN}}$  be the set of  $\lambda \in \Lambda$  satisfying the following three conditions:

(SN1) There exist disjoint isolating neighborhoods  $N_i^\lambda$  in  $N^\lambda$  for  $i = 0, 1$ .

(SN2)  $CH_*(\text{Inv}(N_i^\lambda)) \not\cong 0$ ,  $i = 0, 1$ .

(SN3)  $\{\text{Inv}(N_0^\lambda), \text{Inv}(N_1^\lambda)\}$  is an A-R decomposition of  $\text{Inv}(N^\lambda)$ .

By the robustness of isolating neighborhoods,  $\Lambda_{\text{SN}}$  is an open subset of  $[\lambda_-, \lambda_+]$ .

**Definition 3.2.** Let  $S$  be a compact invariant set in  $X$  and  $x, y \in S$ . An  $\epsilon$ -chain in  $S$  from  $y$  to  $x$  is a sequence

$$\{y = x_1, x_2, \dots, x_{n+1} = x; t_1, t_2, \dots, t_n \mid t_i \geq 1 \text{ for all } i\}$$

which satisfies  $d(x_i \cdot t_i, x_{i+1}) < \epsilon$  for all  $i = 1, 2, \dots, n$ .

**Theorem 3.3.** Let  $N$  be an isolating neighborhood of C-saddle-node type over  $[\lambda_-, \lambda_+]$  and define  $\lambda^* := \inf \Lambda_{\text{SN}}$ . Then the following statements hold.

(1)  $\lambda^* \in [\lambda_-, \lambda_+]$ .

(2) For  $\lambda \in [\lambda_-, \lambda^*]$  and for a Morse decomposition  $\{M(p)\}_{p \in P}$  of  $\text{Inv}(N^\lambda)$ ,

$$CH_*(M(p)) \cong 0, \text{ for all } p \in P.$$

(3) Let  $M_i^{\lambda^*}$  be given by:

$$M_i^{\lambda^*} := \bigcap_{\mu > 0} \overline{\bigcup_{\lambda \in (\lambda^*, \lambda^* + \mu)} M_i^\lambda}, \quad i = 0, 1,$$

where  $\{M_0^\lambda, M_1^\lambda\}$  is an A-R decomposition such that both the attractor  $M_0^\lambda$  and the repeller  $M_1^\lambda$  have non-trivial Conley indices of  $\text{Inv}(N^\lambda)$  for  $\lambda > \lambda^*$ . Then either of the following statements holds:

- $M_0^{\lambda^*} \cap M_1^{\lambda^*} \neq \emptyset$ .
- $M_0^{\lambda^*} \cap M_1^{\lambda^*} = \emptyset$  and for any  $\epsilon > 0$ , there exists an  $\epsilon$ -chain from  $M_0^{\lambda^*}$  to  $M_1^{\lambda^*}$ .

**Definition 3.4.** Let  $N$  be a compact set in  $X \times \Lambda$  and  $\Phi$  be the parameterized semiflow corresponding to the  $\lambda$ -continuous family of flows on  $X$ ,  $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ . We shall say  $\Phi$  undergoes a C-saddle-node bifurcation in  $N$  if  $N$  satisfies all statements in Theorem 3.3 for some  $\lambda^* \in \Lambda$ .

**Remark 3.5.** Notice that the choice of A-R pairs in Theorem 3.3, **(3)**, is not unique and hence each  $M_i^{\lambda^*}$  depends on the choice of A-R pairs. Here we emphasize that a C-saddle-node bifurcation is the generalized notion of the standard saddle-node bifurcation of equilibria in the following sense.

Let  $N$  be an isolating neighborhood of C-saddle-node type over  $[\lambda_-, \lambda_+]$ . At  $\lambda = \lambda_+$ , there exists an A-R pair in  $\text{Inv}(N^{\lambda_+})$  such that both an attractor and the dual repeller have nontrivial Conley indices, which correspond to “saddle” and “node”. By the definition, the dynamics off the attractor and the dual repeller in  $\text{Inv}(N^{\lambda_+})$  is gradient-like. This corresponds to the gradient-like structure around the saddle and the node. In case of the standard saddle-node, two equilibria collapse at the bifurcation point. Such a phenomenon is generalized as follows in case of C-saddle-node. Namely, in Theorem 3.3, **(3)** says that there exists a nontrivial recurrent structure in  $\text{Inv}(N^{\lambda^*})$  which disappears for  $\lambda > \lambda^*$ . Furthermore, **(2)** says that we cannot decompose such a recurrent structure into smaller recurrent ones with nontrivial Conley indices for  $\lambda \leq \lambda^*$ . **(2)** is also the generalized result of the non-existence of equilibria because the Conley index of a hyperbolic fixed point for finite-dimensional flow is non-trivial.

Therefore we can say that the C-saddle-node bifurcation is a “saddle-node” bifurcation of certain recurrent structure of dynamics in the sense of the Conley index.

We show two lemmas before proving the above theorem.

**Lemma 3.6.** *Let  $S$  be a compact invariant set and  $R$  be a compact positively invariant subset of  $S$ . Then a set*

$$\Omega^-(R; S) := \{x \in S \mid \text{For any } \epsilon > 0, \text{ there exists an } \epsilon\text{-chain in } S \text{ from } x \text{ to } y \in R\}$$

*is closed (thus compact) and invariant.*

*Proof.* First we prove that  $\Omega^-(R; S)$  is closed. Let  $\{x^m\}_{m \geq 1}$  be a sequence in  $\Omega^-(R; S)$  which converges to  $x$  as  $m \rightarrow \infty$ . For any  $\epsilon > 0$  and each  $m$ , there exists an  $(\epsilon/2)$ -chain in  $S$  from  $x^m$  to  $y^m \in R$

$$\{x^m = x_0^m, x_1^m, x_2^m, \dots, x_n^m = y^m; t_1^m, \dots, t_n^m \mid t_i^m \geq 1, i = 1, \dots, n\}.$$

By the uniform continuity of  $\varphi(t_1^m, \cdot)$  in  $S$ , for any  $\epsilon > 0$  there is a  $\delta \in (0, \epsilon/2)$  such that  $d(\varphi(t_1^m, a), \varphi(t_1^m, b)) = d(a \cdot t_1^m, b \cdot t_1^m) < (\epsilon/2)$  with  $d(a, b) < \delta$ ,  $a, b \in S$ . If  $m$  is sufficiently large, then  $d(x^m, x) < \delta$ . This implies that  $d(x \cdot t_1^m, x_1^m) \leq d(x \cdot t_1^m, x^m \cdot t_1^m) + d(x^m \cdot t_1^m, x_1^m) < \epsilon$ . Thus we obtain an  $\epsilon$ -chain from  $x$  to  $y^m$ :

$$\{x, x_1^m, x_2^m, \dots, x_n^m = y^m; t_1^m, \dots, t_n^m \mid t_i^m \geq 1, i = 1, \dots, n\},$$

for some  $m$  with  $d(x^m, x) < \delta$ . Since  $\epsilon$  is arbitrary, this implies  $x \in \Omega^-(R; S)$ . Therefore  $\Omega^-(R; S)$  is closed.

Secondly, we prove that  $\Omega^-(R; S)$  is invariant. Let  $x \in \Omega^-(R; S)$  and  $\sigma_x$  be an arbitrary full solution through  $x$  in  $S$ . If

$$\{x = x_1, x_2, \dots, x_{n+1} = y; t_1, t_2, \dots, t_n \mid t_i \geq 1 \text{ for all } i\}$$

is an  $\epsilon$ -chain from  $x$  to  $y \in R$ , a sequence

$$\{\sigma_x(-t), x_2, \dots, x_{n+1} = y; t_1 + t, t_2, \dots, t_n \mid t_i \geq 1 \text{ for all } i\}$$

is obviously an  $\epsilon$ -chain from  $\sigma_x(-t)$  to  $y \in R$ , for  $t \geq 0$ , since  $\sigma_x(-t) \cdot (t_1 + t) = \sigma_x(-t + t_1 + t) = \sigma_x(t_1) = x \cdot t_1$ . This implies that  $\sigma_x(\mathbb{R}_{\leq 0}) \subset \Omega^-(R; S)$  and hence  $\Omega^-(R; S)$  is negatively invariant.

For any  $\epsilon > 0$ , there exists an  $\epsilon$ -chain in  $S$

$$\gamma_\epsilon := \{x = x_1, x_2, \dots, x_{n+1} = y; t_1, t_2, \dots, t_n \mid t_i \geq 1 \text{ for all } i\}$$

from  $x$  to  $y \in R$ . For each  $t \in \mathbb{R}_{\geq 0}$ , by the uniform continuity of  $\varphi(t, \cdot)$  on  $S$ , for any  $\epsilon > 0$ , there exists  $\delta = \delta(t, \epsilon) > 0$  such that if  $d(a, b) < \delta$  then  $d(\varphi(t, a), \varphi(t, b)) = d(a \cdot t, b \cdot t) < \epsilon$ . Thus there exists  $\delta = \delta(t, \epsilon) > 0$  such that  $d(x_i \cdot (t_i + t), x_{i+1} \cdot t) < \epsilon$  with  $d(x_i \cdot t_i, x_{i+1}) < \delta$  for  $i = 1, 2, \dots, n$ . Since  $t$  is arbitrary, for any  $\epsilon > 0$  and  $t \in \mathbb{R}$ , there exists  $\delta > 0$  such that if  $\gamma_\delta$  is a  $\delta$ -chain in  $S$  from  $y$  to  $x$  then there exists an  $\epsilon$ -chain from  $x \cdot t$  to  $y \cdot t \in R$ . This implies  $x \cdot \mathbb{R}_{\geq 0} \subset \Omega^-(R; S)$ . Thus  $\Omega^-(R; S)$  is invariant.  $\square$

**Lemma 3.7.** *Let  $S$  be a compact invariant set for a semiflow  $\varphi$ . Let also  $A$  and  $R$  be disjoint compact positively invariant subsets of  $S$ . If  $\Omega^-(R; S) \cap A = \emptyset$ , then there exists an  $A$ - $R$  decomposition  $\{A_*, R_*\}$  of  $S$  such that  $\Omega^-(A; S) \subset R_*$  and that  $A \subset A_*$ .*

*Proof.* First we claim that there exists an  $\epsilon_0 > 0$  such that  $\Omega_{\epsilon_0}^-(R; S) \cap A = \emptyset$ , where

$$\Omega_{\epsilon_0}^-(R; S) := \{y \in S \mid \text{There exists an } \epsilon_0\text{-chain from } y \text{ to } x \in A\}.$$

Indeed, if not, for any  $\epsilon > 0$ ,  $\Omega_\epsilon^-(R; S) \cap A \neq \emptyset$ . Thus there exists a collection  $\{y_\epsilon\}_{\epsilon > 0} \subset A$  such that  $y_\epsilon \in \Omega_\epsilon^-(R; S)$  for each  $\epsilon$ . Since  $A$  is compact, there exists a subsequence  $\{y_{\epsilon_k}\}_{k \in \mathbb{N}}$  with  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  such that  $y_0 := \lim_{k \rightarrow \infty} y_{\epsilon_k} \in A$ . For any  $\delta > 0$ , there exists  $\bar{\epsilon} \in (0, \delta/2)$  such that  $d(y_\epsilon, y_0) < \delta/2$  for all  $\epsilon \in (0, \bar{\epsilon})$ . Now, by our assumption, there exists a  $\delta/2$ -chain from  $y_\epsilon$  to  $x$  for some  $\epsilon$  for any  $\delta > 0$ . Therefore we obtain a  $\delta$ -chain from  $y_0$  to  $R$ . Since  $\delta > 0$  is arbitrary, this implies that  $y_0 \in \Omega^-(R; S)$ . Hence  $\Omega^-(R; S) \cap A \neq \emptyset$  and this contradicts our assumption.

Now we set  $U := \Omega_{\epsilon_0}^-(R; S)$ .

If  $z \in U$ , there exists an  $\epsilon_0$ -chain  $\{z, x_1, \dots, x_n; t_1, \dots, t_n\}$  from  $z$  to  $R$ . Obviously,  $d(z \cdot t_1, x_1) < \epsilon_0$ . This implies that there exists  $\epsilon_z > 0$  such that  $d(z \cdot t_1, x_1) = \epsilon_z < \epsilon_0$ .

For any  $\eta > 0$ , we can choose  $\delta = \delta(t_1, \eta) > 0$  such that  $d(a \cdot t_1, b \cdot t_1) < \eta$  for all  $a, b \in S$  with  $d(a, b) < \delta$ .

If we let  $\eta_z > 0$  be an arbitrary positive number which satisfies  $\epsilon_z + \eta_z < \epsilon_0$ , then  $d(y \cdot t_1, x_1) < \epsilon_0$  for all  $y \in B(z, \delta(t_1, \eta_z))$ . Therefore,  $U$  is open in  $S$ .

We let  $R_* := \alpha(U)$ . Now we show that  $R_*$  is a repeller in  $S$  containing  $\Omega^-(R; S)$ . Obviously,  $\Omega^-(R; S) \subset U$ . Thus  $\Omega^-(R; S) \subset \bigcup_{x \in U} H(\mathbb{R}_{\geq 0}, x)$ . Taking the closure,  $\Omega^-(R; S) \subset \overline{\bigcup_{x \in U} H(\mathbb{R}_{\geq 0}, x)}$ . Considering the invariant part,  $\Omega^-(R; S) \subset \alpha(U) = R_*$ . Therefore  $R_*$  contains  $\Omega^-(R; S)$ . If  $z \in R_*$ , there exists a point  $x \in U$  and a sequence of real numbers  $\{t^m\}$  such that  $t^m \rightarrow \infty$  and that  $\sigma_x(-t^m) \rightarrow z$ . Now we obtain an  $\epsilon_0$ -chain from  $x$  to  $R$ :

$$\{x = x_1, x_2, \dots, x_{n+1} = y; t_1, t_2, \dots, t_n \mid y \in R, t_i \geq t > 0 \text{ for all } i\}.$$

Now we know that  $d(x \cdot t_1, x_2) := \epsilon_1 < \epsilon_0$ . For any  $\epsilon > 0$ , there exists  $m_0 \in \mathbb{N}$  such that if  $m \geq m_0$  then  $d(\sigma_x(-t^m), z) < \epsilon$ . By the uniform continuity of  $\varphi(t, \cdot)$  on  $S$ , for any  $\epsilon > 0$ , there exists  $\eta = \eta(t, \epsilon) > 0$  such that  $d(a \cdot t, b \cdot t) < \epsilon$  with  $d(a, b) < \eta$ . Taking  $\epsilon > 0$  so small and  $m$  so large so that  $\epsilon + \epsilon_1 < \epsilon_0$  and  $d(z \cdot (t^m + t_1), \sigma_x(-t^m) \cdot (t^m + t_1)) < \epsilon$  with  $d(z, \sigma_x(-t^m)) < \eta(t^m + t_1, \epsilon)$ ,  $d(z \cdot (t^m + t_1), x_2) \leq d(z \cdot (t^m + t_1), \sigma_x(-t^m) \cdot (t^m + t_1)) + d(x_1 \cdot t_1, x_2) < \epsilon + \epsilon_1 < \epsilon_0$  holds. Thus we obtain an  $\epsilon_0$ -chain from  $z$  to  $R$ :

$$\{z, x_2, \dots, x_{n+1} = y; t^m + t_1, t_2, \dots, t_n \mid y \in R, t_i \geq t > 0 \text{ for all } i\}.$$

This implies that  $z \in U$ . Since  $U$  is open, then  $U$  is a neighborhood of  $R_*$ . Therefore  $R_*$  is the repeller we desire. If we let  $A_*$  be the dual attractor of  $R_*$  in  $S$ ,  $A_*$  contains  $A$  and we complete the proof.  $\square$

*Proof of Theorem 3.3.* (1). This follows from the robustness of an isolating neighborhood.

(2). Let  $\{M(p)\}_{p \in P}$  be a Morse decomposition of  $\text{Inv}(N^\lambda)$  for  $\lambda \in [\lambda_-, \lambda^*]$ , where  $P = (P, <)$  is a partially ordered set. We assume that there exists  $p \in P$  such that  $CH_*(M(p)) \neq 0$ . Let

$$p_0 := \min\{p \in P \mid CH_*(M(p)) \neq 0\}.$$

If  $I := \{q \in P \mid q < p_0\} \cup \{p_0\}$  and  $J := P \setminus I$ , then  $I$  and  $J$  are adjacent intervals and  $I \cup J = P$ . Moreover,  $\{M(I), M(J)\}$  is an A-R decomposition of  $\text{Inv}(N^\lambda)$ .

We claim that the Conley indices of  $M(I)$  and  $M(J)$  are non-trivial. Since  $p_0$  is the maximal element of  $I$  for the order  $<$ ,  $I \setminus \{p_0\}$  is also an interval in  $<$ . Since  $I \setminus \{p_0\}$  and  $\{p_0\}$  are adjacent,  $\{M(I \setminus \{p_0\}), M(p_0)\}$  is an A-R pair of  $M(I)$ . Therefore there exists a long exact sequence

$$\cdots \rightarrow CH_n(M(I \setminus \{p_0\})) \rightarrow CH_n(M(I)) \rightarrow CH_n(M(p_0)) \xrightarrow{\partial} CH_{n-1}(M(I \setminus \{p_0\})) \rightarrow \cdots \quad (1)$$

By the induction for the number of elements of  $I \setminus \{p_0\}$ , we obtain

$$CH_*(M(I \setminus \{p_0\})) = 0$$



by using long exact sequences. Therefore, by the long exact sequence (1), we obtain

$$CH_*(M(I)) = CH_*(M(p_0)) \neq 0.$$

Moreover, since  $\{M(I), M(J)\}$  is an A-R decomposition of  $\text{Inv}(N^\lambda)$ , there exists a long exact sequence

$$\cdots \rightarrow CH_n(M(I)) \rightarrow CH_n(\text{Inv}(N^\lambda)) \rightarrow CH_n(M(J)) \xrightarrow{\partial} CH_{n-1}(M(I)) \rightarrow \cdots \quad (2)$$

This contradicts the definition of  $\lambda^*$ .

(3). We remark that  $M_i^{\lambda^*}$  is positively invariant. Indeed, if we assume  $x \in M_i^{\lambda^*}$ , then for all  $\mu > 0$ ,  $x \in \overline{\bigcup_{\lambda \in (\lambda^*, \lambda^* + \mu)} M_i^\lambda}$ . Since the set  $\bigcup_{\lambda \in (\lambda^*, \lambda^* + \mu)} M_i^\lambda$  is closed, there exists a sequence  $\{x_n\}_{n \geq 1} \subset \bigcup_{\lambda \in (\lambda^*, \lambda^* + \mu)} M_i^\lambda$  that converges to  $x$  as  $n \rightarrow \infty$ . For each  $n \geq 1$ ,  $x_n$  belongs to  $M_i^{\lambda^* + \mu_n}$  for some  $\mu_n \in (0, \mu)$ . Thus  $x_n \cdot \mathbb{R} \subset M_i^{\lambda^* + \mu_n}$  because  $M_i^{\lambda^* + \mu_n}$  is invariant (especially, positively invariant) for each  $n$ . Namely,  $x_n \cdot t \in M_i^{\lambda^* + \mu_n}$  for each  $n$  and for all  $t \in \mathbb{R}_{\geq 0}$ . Then the sequence  $\{x_n \cdot t\}$  converges to  $x \cdot t$  by the continuity of the semiflow. Therefore  $x \cdot \mathbb{R}_{\geq 0} \subset \overline{\bigcup_{\lambda \in (\lambda^*, \lambda^* + \mu)} M_i^\lambda}$ . Since  $\mu$  is arbitrary,

$$M_i^{\lambda^*} = \bigcap_{\mu > 0} \overline{\bigcup_{\lambda \in (\lambda^*, \lambda^* + \mu)} M_i^\lambda} \text{ is positively invariant.}$$

Now we assume that  $M_0^{\lambda^*} \cap M_1^{\lambda^*} = \emptyset$ . We define

$$\Omega^-(M_1^{\lambda^*}) = \Omega^-(M_1^{\lambda^*}; \text{Inv}(N^{\lambda^*})) := \{y \in \text{Inv}(N^{\lambda^*}) \mid$$

For any  $\epsilon > 0$ , there exists an  $\epsilon$ -chain in  $\text{Inv}(N^{\lambda^*})$  from  $y$  to  $x \in M_1^{\lambda^*}\}$ .

By Lemma 3.6,  $\Omega^-(M_1^{\lambda^*})$  is compact and invariant.

If  $\Omega^-(M_1^{\lambda^*}) \cap M_0^{\lambda^*} = \emptyset$ , by Lemma 3.7, there exists an A-R decomposition  $\{A_*, R_*\}$  of  $\text{Inv}(N^{\lambda^*})$  such that  $R_*$  contains  $\Omega^-(M_1^{\lambda^*})$  and that  $A_*$  contains  $M_0^{\lambda^*}$ . If we let  $N_{R_*}$  be an isolating neighborhood of  $R_*$  and  $N_{A_*}$  be an isolating neighborhood of  $A_*$ , both neighborhoods are isolating neighborhoods for  $\lambda \in (\lambda^*, \lambda^* + \delta)$  for small  $\delta > 0$  by the robustness of isolating neighborhoods.

If we fix  $\lambda \in (\lambda^*, \lambda^* + \delta)$ , then  $\text{Inv}(N_{A_*}, \varphi^\lambda) = M_0^\lambda$  and  $\text{Inv}(N_{R_*}, \varphi^\lambda) = M_1^\lambda$  because  $\{M_0^\lambda, M_1^\lambda\}$  is an A-R decomposition in  $\text{Inv}(N^\lambda)$ . Since both the Conley indices of  $M_0^\lambda$  and  $M_1^\lambda$  are non-trivial,  $\{\text{Inv}(N_{A_*}), \text{Inv}(N_{R_*})\}$  is an A-R decomposition of  $\text{Inv}(N^\lambda)$  which satisfies

$$CH_*(\text{Inv}(N_{A_*}), \varphi^\lambda) \neq 0, \quad CH_*(\text{Inv}(N_{R_*}), \varphi^\lambda) \neq 0,$$

by the robustness of isolating neighborhoods, where  $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ . This contradicts the definition of  $\lambda^*$ . Therefore  $\Omega^-(M_1^{\lambda^*}) \cap M_0^{\lambda^*} \neq \emptyset$ . □

By the above theorem, we know immediately that  $\text{Inv}(N^{\lambda^*})$  cannot be decomposed into a nontrivial A-R pair which separates  $M_0^{\lambda^*}$  and  $M_1^{\lambda^*}$ .

### 3.2 Pitchfork type

Now we introduce a weaker notion of pitchfork bifurcation in the view of Conley index theory. We assume that the group  $\mathbb{Z}_2$  acts on  $X$  and satisfies

$$\varphi^\lambda(t, gu) = g\varphi^\lambda(t, u), \quad (3)$$

for all  $g \in \mathbb{Z}_2$ ,  $t \in \mathbb{R}_{\geq 0}$ ,  $u \in X$  and  $\lambda \in \Lambda$ . Assume that  $g \in \mathbb{Z}_2$  is the non-trivial element unless we say otherwise.

**Definition 3.8.** Let  $N$  be a connected isolating neighborhood for  $\Phi$  such that  $N^\lambda$  is connected for each  $\lambda$  and that the restriction of the semiflow  $\varphi^\lambda|_{\text{Inv}(N^\lambda)}$  is a flow.. We say  $N$  is of *C-pitchfork type* over  $[\lambda_-, \lambda_+]$  if the following conditions are satisfied:

(CPF1)

$$CH_*(\text{Inv}(N^{\lambda_-}), \varphi^{\lambda_-}) \not\cong 0$$

and there is not a Morse decomposition of  $\text{Inv}(N^{\lambda_-})$  containing  $\mathbb{Z}_2$ -asymmetric component.

Moreover, there exist mutually disjoint isolating neighborhoods  $N_0^{\lambda_+}$ ,  $N_1^{\lambda_+}$  and  $N_2^{\lambda_+}$  such that

(CPF2) A collection of isolated invariant sets  $\{\text{Inv}(N_0^{\lambda_+}), \text{Inv}(N_1^{\lambda_+}), \text{Inv}(N_2^{\lambda_+})\}$  forms a Morse decomposition of  $\text{Inv}(N^{\lambda_+})$  with one of the following admissible orderings:

$$\text{(S)} \quad \text{Inv}(N_0^{\lambda_+}) < \text{Inv}(N_2^{\lambda_+}), \quad \text{Inv}(N_1^{\lambda_+}) < \text{Inv}(N_2^{\lambda_+}),$$

$$\text{(U)} \quad \text{Inv}(N_0^{\lambda_+}) > \text{Inv}(N_2^{\lambda_+}), \quad \text{Inv}(N_1^{\lambda_+}) > \text{Inv}(N_2^{\lambda_+}).$$

See Remark 2.11 for the definition of an ordering on Morse decompositions.

(CPF3)  $\text{Inv}(N_j^{\lambda_+})$ ,  $j = 0, 1, 2$ , satisfy the following conditions:

$$\text{Inv}(N_i^{\lambda_+}) \cap \text{Fix}(\mathbb{Z}_2) = \emptyset \quad \text{and} \quad g\text{Inv}(N_i^{\lambda_+}) = \text{Inv}(N_{1-i}^{\lambda_+}), \quad i = 0, 1,$$

$$g\text{Inv}(N_2) = \text{Inv}(N_2),$$

where  $\text{Fix}(\mathbb{Z}_2)$  is a subset of  $X$  defined by  $\text{Fix}(\mathbb{Z}_2) = \{x \in X \mid gx = x \text{ for all } g \in \mathbb{Z}_2\}$ .

(CPF4)

$$CH_*(\text{Inv}(N_i^{\lambda_+}), \varphi^{\lambda_+}) \not\cong 0, \quad i = 0, 1.$$

(CPF5) There exists a  $\mathbb{Z}_2$ -symmetric isolated invariant set  $S \subset \text{Inv}(N_2^{\lambda_+})$  with nontrivial Conley index.

We shall say  $N$  be an isolating neighborhood of C-pitchfork type (over  $[\lambda_-, \lambda_+]$ ) with *type-S admissible ordering* if (S) in the above theorem holds. Similarly, we shall say  $N$  be an isolating neighborhood of C-pitchfork type (over  $[\lambda_-, \lambda_+]$ ) with *type-U admissible ordering* if (U) in the above theorem holds.

We remark that if  $N$  is of C-pitchfork type over  $[\lambda_-, \lambda_+]$ ,  $N$  also satisfies

$$CH_*(\text{Inv}(N^{\lambda_+}), \varphi^{\lambda_+}) = CH_*(\text{Inv}(N^{\lambda_-}), \varphi^{\lambda_-}),$$

by the continuation property.

Here we shall capture the behavior of isolated invariant sets in a C-pitchfork type isolating neighborhood. Let  $\Lambda_{\text{PF}}$  be the set of  $\lambda$  such that each  $M_i^\lambda$  ( $i = 0, 1, 2$ ) is nonempty invariant subset in  $N^\lambda$  and there exist subsets  $N_0^\lambda$ ,  $N_1^\lambda$  and  $N_2^\lambda$  in  $N^\lambda$  satisfying the following conditions.

(PF1)  $N_i^\lambda$  is an isolating neighborhood including  $M_i^\lambda$  (not necessary  $M_i^\lambda = \text{Inv}(N_i^\lambda)$ ) for  $i = 0, 1, 2$ .

(PF2)  $\{\text{Inv}(N_0^\lambda), \text{Inv}(N_1^\lambda), \text{Inv}(N_2^\lambda)\}$  satisfies the condition (CPF2)-(CPF5).

By the robustness of the isolating neighborhoods,  $\Lambda_{\text{PF}}$  is an open subset of  $[\lambda_-, \lambda_+]$ .

**Theorem 3.9.** Let  $N$  be an isolating neighborhood of C-pitchfork type over  $[\lambda_-, \lambda_+]$ , a collection

$$\mathcal{M}_\lambda := \{M_0^\lambda, M_1^\lambda, M_2^\lambda\}$$

be an arbitrary Morse decomposition of  $\text{Inv}(N^\lambda)$ ,  $\lambda \in [\lambda_-, \lambda_+]$  which satisfies (CPF2) and (CPF3), and define  $\lambda^* := \inf \Lambda_{\text{PF}}$ . Then the following statements hold.

(1)  $\lambda^* \in [\lambda_-, \lambda_+]$ .

(2) For  $\lambda \in [\lambda_-, \lambda^*]$  and for  $\mathcal{M}_\lambda$  which also satisfies (CPF5), we have

$$CH_*(M_0^\lambda) = CH_*(M_1^\lambda) \cong 0.$$

If  $\{M(p)\}_{p \in P}$  is a Morse decomposition of  $M_0^\lambda$  for  $\lambda \in [\lambda_-, \lambda^*]$ , then

$$CH_*(M(p)) \cong 0, \text{ for all } p \in P.$$

Moreover, if  $A$  is an attractor in  $\text{Inv}(N^\lambda)$  such that  $gA \cap A = \emptyset$ , then

$$CH_*(A) \cong 0.$$

Similarly, if  $R$  is a repeller in  $\text{Inv}(N^\lambda)$  such that  $gR \cap R = \emptyset$ , then

$$CH_*(R) \cong 0.$$

(3) Let  $M_i^{\lambda^*}$  be given by:

$$M_i^{\lambda^*} := \bigcap_{\epsilon > 0} \overline{\bigcup_{\lambda \in (\lambda^*, \lambda^* + \epsilon)} M_i^\lambda}, \quad i = 0, 1, 2,$$

where  $M_i^\lambda = \text{Inv}(N_i^\lambda)$ ,  $i = 0, 1, 2$  are isolated invariant sets which satisfies (CPF2-4) for  $\lambda > \lambda^*$ . Then one of the following statements holds:

- $\bigcap_{i=0,1,2} M_i^{\lambda^*} \neq \emptyset$ .
  - $M_0^{\lambda^*} \cap M_1^{\lambda^*} \neq \emptyset$  and  $M_i^{\lambda^*} \cap M_2^{\lambda^*} = \emptyset$ ,  $i = 0, 1$ .
  - $M_0^{\lambda^*} \cap M_1^{\lambda^*} = \emptyset$  and  $M_i^{\lambda^*} \cap M_2^{\lambda^*} \neq \emptyset$ ,  $i = 0, 1$ .
  - If  $\lambda > \lambda_*$  is sufficiently close to  $\lambda_*$  and  $M_i^{\lambda^*}$ ,  $i = 0, 1, 2$ , are mutually disjoint, there exists an  $\epsilon$ -chain:
    - from  $M_i^{\lambda^*}$ ,  $i = 0, 1$ , to  $M_2^{\lambda^*}$  if  $\mathcal{M}_\lambda$  has type-S ordering,
    - from  $M_2^{\lambda^*}$  to  $M_i^{\lambda^*}$ ,  $i = 0, 1$ , if  $\mathcal{M}_\lambda$  has type-U ordering or
    - from  $M_0^{\lambda^*}$  to  $M_1^{\lambda^*}$  and from  $M_0^{\lambda^*}$  to  $M_1^{\lambda^*}$ ,
- for any  $\epsilon > 0$ .

**Definition 3.10.** Let  $N$  be a compact set in  $X \times \Lambda$  and  $\Phi$  be the parameterized semiflow corresponding to the  $\lambda$ -continuous family of flows on  $X$ ,  $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ . We shall call  $\Phi$  undergoes a C-pitchfork bifurcation in  $N$  if  $N$  holds all statements in Theorem 3.9 for some  $\lambda^* \in \Lambda$ .

**Remark 3.11.** Notice that the choice of  $\mathcal{M}_\lambda$  in Theorem 3.9, (3), is not unique and hence each  $M_i^{\lambda^*}$  depends on the choice of  $\mathcal{M}_\lambda$  for  $\lambda (> \lambda_*)$  which is sufficiently close to  $\lambda_*$ . We can say that the C-pitchfork bifurcation is a generalization of standard pitchfork in the similar sense to C-saddle-node (see Remark 3.5).

Namely, we can say that the C-pitchfork bifurcation is a “pitchfork” bifurcation of certain recurrent structure of dynamics in the sense of the Conley index.

*Proof of Theorem 3.9.* (1). By the robustness of isolating neighborhoods and Morse decompositions of an isolated invariant set,  $\lambda^* < \lambda_+$ .

(2). Let  $\mathcal{M}_\lambda$  be a Morse decomposition (in the statement of theorem) for  $\lambda \in [\lambda_-, \lambda^*]$  with type-S admissible ordering which satisfies (CPF5). We immediately obtain the result

$$CH_*(M_0^\lambda) = CH_*(M_1^\lambda) \cong 0,$$

by the definition of  $\lambda^*$ .

Let  $\{M^{(0)}(p)\}_{p \in P}$  be a Morse decomposition of  $M_0^\lambda$ ,  $\lambda \in [\lambda_-, \lambda^*]$ . We assume that there exists  $p \in P$  such that  $CH_*(M^{(0)}(p)) \neq 0$ . Let

$$p_0 := \min\{p \in P \mid CH(M^{(0)}(p)) \neq 0\}.$$

If  $I := \{q \in P \mid q < p_0\} \cup \{p_0\}$  and  $J := P \setminus I$ , then  $I$  and  $J$  are adjacent intervals and  $I \cup J = P$ . Hence  $\{M^{(0)}(I), M^{(0)}(J)\}$  is a A-R decomposition of  $\text{Inv}(N^\lambda)$ .

We consider the Conley indices of  $M^{(0)}(I)$  and  $M^{(0)}(J)$ . Since  $p_0$  is the maximal element of  $I$  for the order  $<$ ,  $I \setminus \{p_0\}$  is also an interval in  $<$ . Since  $I \setminus \{p_0\}$  and  $\{p_0\}$  are adjacent,  $\{M^{(0)}(I \setminus \{p_0\}), M^{(0)}(p_0)\}$  is an A-R pair of  $M^{(0)}(I)$ . Therefore there exists a long exact sequence

$$\cdots \rightarrow CH_n(M^{(0)}(I \setminus \{p_0\})) \rightarrow CH_n(M^{(0)}(I)) \rightarrow CH_n(M^{(0)}(p_0)) \xrightarrow{\partial} CH_{n-1}(M^{(0)}(I \setminus \{p_0\})) \rightarrow \cdots \quad (4)$$

By the induction on the number of elements of  $I \setminus \{p_0\}$ , we obtain

$$CH_*(M^{(0)}(I \setminus \{p_0\})) \cong 0$$

by using long exact sequences. Therefore, by the long exact sequence (4), we obtain

$$CH_*(M^{(0)}(I)) = CH_*(p_0) \cong 0.$$

Moreover, since  $\{M^{(0)}(I), M^{(0)}(J)\}$  is an A-R decomposition of  $M_0^\lambda$ , there exists a long exact sequence

$$\cdots \rightarrow CH_n(M^{(0)}(I)) \rightarrow CH_n(M_0^\lambda) \rightarrow CH_n(M^{(0)}(J)) \xrightarrow{\partial} CH_{n-1}(M^{(0)}(I)) \rightarrow \cdots \quad (5)$$

Therefore we obtain

$$CH_n(M^{(0)}(J)) = CH_{n-1}(M^{(0)}(I))$$

and hence  $CH_*(M^{(0)}(J)) \neq 0$  because  $CH_*(M_0^\lambda) = 0$ . We remark that  $M^{(0)}(I)$  is an attractor in  $\text{Inv}(N^\lambda)$  because  $M^{(0)}(I)$  is an attractor in  $M_0^\lambda$  and  $M_0^\lambda$  is an attractor in  $\text{Inv}(N^\lambda)$ .

If  $\{M^{(0)}(p)\}_{p \in P}$  is a Morse decomposition of  $M_0^\lambda$ , then

$$\{M^{(1)}(p) = gM^{(0)}(p)\}_{p \in P}$$

is a Morse decomposition of  $M_1^\lambda$ . Moreover, we know that  $gM^{(0)}(I)$  is an attractor in  $\text{Inv}(N^\lambda)$  and has non-trivial Conley index by the above consideration. Since  $M_0^\lambda$  and  $M_1^\lambda$  are unrelated for the order  $<$ ,  $M^{(0)}(I)$  and  $gM^{(0)}(I)$  are also unrelated for the order  $<$ . Therefore

$$\{M^{(0)}(I), gM^{(0)}(I), \hat{M}\}$$

is a Morse decomposition of  $\text{Inv}(N^\lambda)$  with type-S ordering, where  $\hat{M} = M_2^\lambda \cup M^{(0)}(J) \cup gM^{(0)}(J) \cup C(M^{(0)}(J), M_2^\lambda) \cup C(gM^{(0)}(J), M_2^\lambda)$ . By the robustness of isolating neighborhoods, there exist isolating neighborhoods  $N^{(0)}(I)$  of  $M^{(0)}(I)$ ,  $gN^{(0)}(I)$  of  $gM^{(0)}(I)$  and  $\hat{N}$  of  $\hat{M}$  and there exists  $\delta > 0$  such that

$$\{\text{Inv}(N^{(0)}(I)), \text{Inv}(gN^{(0)}(I)), \text{Inv}(\hat{M})\}$$

is a Morse decomposition of  $\text{Inv}(N^\eta)$  with the ordering

$$\text{Inv}(N^{(0)}(I)) < \text{Inv}(\hat{M}), \text{Inv}(gN^{(0)}(I)) < \text{Inv}(\hat{M}),$$

which satisfies (CPF3) and (CPF5), for  $\eta \in (\lambda - \delta, \lambda + \delta)$ . Since  $M^{(0)}(I)$  and  $gM^{(0)}(I)$  have non-trivial Conley indices, this contradicts the definition of  $\lambda^*$ .

If  $\mathcal{M}_\lambda$  is a Morse decomposition (in the statement of theorem) for  $\lambda \in [\lambda_-, \lambda^*]$  with type-U admissible ordering which satisfies (CPF5), we can construct a Morse decomposition

$$\{\check{M}, M^{(0)}(J), gM^{(0)}(J)\}$$

of  $\text{Inv}(N^\lambda)$  with type-U ordering, where  $\check{M} = M_2^\lambda \cup M^{(0)}(I) \cup gM^{(0)}(I) \cup C(M^{(0)}(I), M_2^\lambda) \cup C(gM^{(0)}(I), M_2^\lambda)$ . We know that both  $M^{(0)}(J)$  and  $gM^{(0)}(J)$  have non-trivial Conley indices by the above consideration. Thus we can prove the contradiction by the above consideration.

**(3)**. We know that  $M_i^{\lambda^*}$  is positively invariant by the same argument as in **(3)** of Theorem 3.3.

Now we prove if  $M_0^{\lambda^*}, M_1^{\lambda^*}$  and  $M_2^{\lambda^*}$  are mutually disjoint, then the last statement holds.

Here we consider only the case that  $\mathcal{M}_\lambda$ ,  $\lambda > \lambda^*$ , has type-U admissible ordering, that is,

$$M_0^\lambda > M_2^\lambda, \quad M_1^\lambda > M_2^\lambda.$$

We define  $\Omega^-(M_i^{\lambda^*})$ ,  $i = 0, 1$ , as the proof of Theorem 3.3. First, we consider the case  $\Omega^-(M_0^{\lambda^*}) \cap (M_1^{\lambda^*} \cup M_2^{\lambda^*}) = \emptyset$ . By  $\mathbb{Z}_2$ -symmetry, this implies that  $\Omega^-(M_0^{\lambda^*})$ ,  $\Omega^-(M_1^{\lambda^*})$  and  $M_2^{\lambda^*}$  are mutually disjoint.

In this case, by Lemma 3.7, there exist repellers  $R_i^*$ ,  $i = 0, 1$ , such that each  $R_i^*$  contains  $\Omega^-(M_i^{\lambda^*})$  and that  $R_0^*$ ,  $R_1^*$  and  $M_2^{\lambda^*}$  are mutually disjoint.

If we let  $A_2^*$  be the dual attractor of  $R_0^* \cup R_1^*$  in  $\text{Inv}(N^{\lambda^*})$ , then  $A_2^*$  contains  $M_2^{\lambda^*}$ . There exist isolating neighborhoods  $N_0^*$ ,  $N_1^*$  and  $N_2^*$  of  $R_0^*$ ,  $R_1^*$  and  $A_2^*$ , respectively, such that they are mutually disjoint.

By the robustness of the isolating neighborhoods, there exists  $\delta > 0$  such that  $N_i^*$ ,  $i = 0, 1, 2$ , are isolating neighborhoods in  $\text{Inv}(N^\lambda)$ , where  $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ . If we fix  $\lambda \in (\lambda^*, \lambda^* + \delta)$ , then  $\text{Inv}(N_i^*, \varphi^\lambda) = M_i^\lambda$ ,  $i = 0, 1, 2$ , because  $\{M_0^\lambda, M_1^\lambda, M_2^\lambda\}$  is an Morse decomposition in  $\text{Inv}(N^\lambda)$ . Since both the Conley indices of  $M_0^\lambda$  and  $M_1^\lambda$  are non-trivial,

$$CH_*(\text{Inv}(N_i^*, \varphi^{\lambda^*})) \not\cong 0, \quad i = 0, 1$$

by the continuation property of the Conley indices. Moreover, by the robustness of the isolating neighborhoods,  $\{\text{Inv}(N_0^*), \text{Inv}(N_1^*), \text{Inv}(N_2^*)\}$  is the Morse decomposition of  $\text{Inv}(N^\lambda)$  which satisfies (CPF2-4), where  $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$ . This contradicts the definition of  $\lambda^*$ . □

Next we show that any given Morse decomposition of a  $\mathbb{Z}_2$ -symmetric isolated invariant set can be replaced by one whose each Morse set  $M$  is either  $\mathbb{Z}_2$ -symmetric or satisfies  $M \cap gM = \emptyset$ ,  $g \in \mathbb{Z}_2 \setminus \{id.\}$ .

**Proposition 3.12.** *If  $S$  is a  $\mathbb{Z}_2$ -symmetric isolated invariant set and if*

$$\{M(i)\}_{i \in \{1, \dots, n\}}$$

*is a Morse decomposition of  $S$  with the admissible ordering  $<$ , then a collection of isolated invariant sets*

$$\mathcal{M} := \{\tilde{M}_{ij} \mid i, j \in \{1, \dots, n\}, \tilde{M}_{ij} \neq \emptyset\}, \quad \text{where } \tilde{M}_{ij} = M(i) \cap gM(j), \quad i, j \in \{1, \dots, n\},$$

*is a Morse decomposition of  $S$  with the following property:*

$$\text{If } \tilde{M}_{ij}, \quad i, j \in \{1, \dots, n\} \text{ is } \mathbb{Z}_2\text{-asymmetric, then } \tilde{M}_{ij} \cap g\tilde{M}_{ij} = \emptyset \quad (6)$$

*for which an admissible ordering  $<_{\mathcal{M}}$  of  $\mathcal{M}$  can be given by*

- $\tilde{M}_{ik} <_{\mathcal{M}} \tilde{M}_{jl}$  if  $i < j$  and  $k < l$ .
- $\tilde{M}_{ik} <_{\mathcal{M}} \tilde{M}_{jk}$  if  $i < j$ .
- $\tilde{M}_{ik} <_{\mathcal{M}} \tilde{M}_{il}$  if  $k < l$ .

*Proof.* The relation (6) follows from the fact that  $(\tilde{M}_{ij}) \subset M(i)$ ,  $g(\tilde{M}_{ij}) \subset M(j)$  and  $M(i) \cap M(j) = \emptyset$ . We shall prove the following statements:

- $<_{\mathcal{M}}$  is a partial order.
- $<_{\mathcal{M}}$  contains the flow order of  $\mathcal{M}$ . Namely, the existence of a connecting orbit from  $\tilde{M}_{jl}$  to  $\tilde{M}_{ik}$  implies  $\tilde{M}_{ik} <_{\mathcal{M}} \tilde{M}_{jl}$ .

The first statement is trivial.

We assume that there exists a connecting orbit from  $\tilde{M}_{jl}$  to  $\tilde{M}_{ik}$ . Thus there exists a connecting orbit from  $M(j)$  to  $M(i)$  and from  $gM(l)$  to  $gM(k)$ , namely, from  $M(l)$  to  $M(k)$ . This implies  $i < j$  and  $k < l$ . Hence  $\tilde{M}_{ik} <_{\mathcal{M}} \tilde{M}_{jl}$ .

Therefore  $\mathcal{M}$  is a Morse decomposition with an admissible ordering  $<_{\mathcal{M}}$  which satisfies (6). □

This proposition presents that, for a Morse decomposition of a  $\mathbb{Z}_2$ -symmetric isolated invariant set, we can reconstruct a Morse decomposition of  $S$  which satisfies the assumption for a Morse decomposition of **(2)** in Theorem 3.9.

Let  $\mathcal{M}$  be a Morse decomposition of  $S$  which we have obtained in Proposition 3.12. We define a subset  $I_A$  in  $\{1, \dots, n\} \times \{1, \dots, n\}$  as follows:

$$I_A := \{(i, j) \text{ with } i <_{\mathbb{Z}} j \mid \tilde{M}_{ij} \text{ is } \mathbb{Z}_2\text{-asymmetric}\},$$

where  $<_{\mathbb{Z}}$  is the ordinary ordering on  $\mathbb{Z}$ .

If there exists  $I_0 \subset I_A$  such that  $I_0$  is an attracting interval in  $<_{\mathcal{M}}$  (see [17]), then

$$M(I_0) := \{\tilde{M}_{ij} \mid (i, j) \in I_0\} \cup \{C(\tilde{M}_{ij}, \tilde{M}_{kl}; S) \mid (i, j), (k, l) \in I_0\}$$

is an attractor in  $S$ . Since  $S$  is  $\mathbb{Z}_2$ -symmetric,  $gM(I_0)$  is another attractor in  $S$ . Therefore, if  $R$  is the dual repeller of  $M(I_0) \cup gM(I_0)$  in  $S$ , then

$$\{M(I_0), gM(I_0), R\}$$

is a Morse decomposition of  $S$  which satisfies the assumption of **(2)** in Theorem 3.9. By similar arguments, we reconstruct the Morse decomposition  $\{M'_0, M'_1, M'_2\}$  of  $S$  with the following admissible ordering:

$$M'_0 > M'_2, M'_1 > M'_2.$$

We end this section providing simple examples of C-pitchfork bifurcation. We consider the following 1-dimensional systems :

$$\dot{x} = f_1(\lambda, x) := \lambda x - x^3, \quad (7)$$

$$\dot{x} = f_2(\lambda, x) := \lambda x - x^5, \quad (8)$$

$$\dot{x} = f_3(\lambda, x) := \begin{cases} (\lambda + 1)x - x^3 & \lambda > 0 \text{ or } \lambda = 0, |x| > 1 \\ 0 & \lambda = 0 \text{ and } |x| \leq 1 \\ \lambda x - x^3 & \lambda < 0 \end{cases}. \quad (9)$$

and

$$\dot{x} = f_4(\lambda, x) := (\lambda - 1)x + 8x^3 - 16x^5, \quad (10)$$

Set  $N \subset \mathbb{R} \times \mathbb{R}$  as  $N := [-1.1, 1.1] \times [-2, 2]$ . Obviously,  $\text{Inv}(N \cap (\{-1.1\} \times \mathbb{R})) = \{0\}$  in any cases. Easy computations yields that  $\text{Inv}(N \cap (\{+1.1\} \times \mathbb{R}))$  contains exactly three hyperbolic fixed points such that one of them is unstable and the rest are stable. We also know that each vector field on  $\partial(N \cap (\{\lambda\} \times \mathbb{R}))$  for each  $\lambda \in [-1.1, 1.1]$  in any cases. Therefore (7), (8), (9) and (10) possess the same isolating neighborhoods of C-pitchfork type. A different type of bifurcation occurs in each system. In (7), the standard pitchfork bifurcation occurs. In (8), the bifurcation is a *degenerate* pitchfork. In (9), the vector field is not even smooth. In (10), a (subcritical) pitchfork and a saddle-node bifurcation occur in  $N$ . However, in terms of the change of gradient-like structure of dynamics, the same type of bifurcations occur at  $\lambda = 0$ .

In (7) and (8), there exist trivial  $\epsilon$ -chains from the upper limit of stable fixed points to the upper limit of the unstable fixed point.

In (9), taking  $M_0^\lambda := \{\sqrt{\lambda + 1}\}$ ,  $M_1^\lambda := \{-\sqrt{\lambda + 1}\}$  and  $M_2^\lambda := \{0\}$ , we know that there exist  $\epsilon$ -chains from  $M_0^0 := \{1\}$ ,  $M_1^0 := \{-1\}$  to  $M_2^0 := \{0\}$ , although  $M_0^\lambda$  and  $M_1^\lambda$  are attractors for  $\lambda > 0$  and  $M_2^\lambda$  is a repeller for  $\lambda > 0$ .

In (10), taking  $M_0^\lambda := \{\sqrt{1 + \sqrt{\lambda}}/2\}$ ,  $M_1^\lambda := \{-\sqrt{1 + \sqrt{\lambda}}/2\}$  and

$$M_2^\lambda := \{0\} \cup \left\{ \frac{\sqrt{1 - \sqrt{\lambda}}}{2} \right\} \cup \left\{ \frac{-\sqrt{1 - \sqrt{\lambda}}}{2} \right\} \cup \{\text{connecting orbits between them}\}, \quad (11)$$

we know that a collection of invariant sets  $\{M_0^\lambda, M_1^\lambda, M_2^\lambda\}$  is a Morse decomposition of  $\text{Inv}(N^\lambda)$  (for  $0 < \lambda < 1$ ) and that  $M_0^0 \cap M_2^0 \neq \emptyset$  and  $M_1^0 \cap M_2^0 \neq \emptyset$ . In other words, there exist (trivial)  $\epsilon$ -chains from the limit of attractors  $M_0^0$  and  $M_1^0$  to the limit of repeller  $M_2^0$ .

Therefore we know that there exists a new recurrent structure at  $\lambda = 0$  which disappears for  $\lambda > 0$  and the behavior of recurrent structure is like a “pitchfork” in any case. We identify such behaviors with a “C-pitchfork bifurcation” in our view.

## 4 Overview of numerical verification methods for application

In this section, we review pioneering verification methods for capturing invariant objects in infinite dimensional dynamics with rigorous numerics, as preliminaries of our application in the next section. For details, see corresponding references.

### (1) A self-consistent a priori bound [22].

Zgliczyński and Mischaikow have proposed a verification method to prove the existence of equilibria for parabolic PDEs in [22]. We consider a vector field

$$\frac{du}{dt} = F(u) \quad (12)$$

in a Hilbert space  $H$  and a pair  $(W, \{a_k^\pm\}_{k=0,1,\dots})$  of a compact set  $W$  in  $m$ -dimensional subspace of  $H$  for  $m \in \mathbb{N}$  and a sequence of real numbers  $\{a_k^\pm\}$  with  $a_k^- < a_k^+$  for each  $k \in \mathbb{N} \cap \{0\}$ . Here we assume that the Hilbert space  $H$  is separable and spanned by a countable number of functions  $\{\varphi_k\}_{k=0,1,\dots}$  as an orthonormal basis. We also assume that the pair  $(W, \{a_k^\pm\})$  defines a compact set

$$Z := W \times \prod_{k \geq m} [a_k^-, a_k^+] \quad (13)$$

in  $H$  and that  $F$  is bounded and continuous in  $Z$ , where each interval  $[a_k^-, a_k^+]$  is regarded as an interval in the one-dimensional subspace  $\text{span}\{\varphi_k\}$ . The pair  $(W, \{a_k^\pm\})$  gives us appropriate properties and a bound of solutions and vector fields. We say that such a pair is a *self-consistent a priori bound* for vector field (12).

Their approach to find an equilibrium for (12) is based on the construction of a sequence  $\{z_n\}_{n \geq m}$  which consists of equilibria of finite dimensional Galerkin approximated dynamical system

$$\frac{dp_n}{dt} = P_n F(p_n, q_n) \quad (14)$$

for  $n \geq m$ , where  $P_n : H \rightarrow X_n := \text{span}\{\varphi_0, \dots, \varphi_{n-1}\}$  is the orthogonal projection onto the  $n$ -dimensional subspace  $X_n$ ,  $p_n = P_n u$  and  $q_n = (I - P_n)u$  are corresponding projections of an element  $u \in H$ .

We obtain a rigorous stationary solution  $z_*$  of (12) as follows. First we construct a self-consistent a priori bound  $(W, \{a_k^\pm\})$  so that the set

$$W \times \prod_{k=m}^n [a_k^-, a_k^+]$$

contains an isolating block  $B_n$  of (14) for each  $n \geq m$ . We use rigorous numerics here to construct a bound which possesses such an appropriate structure for isolation, which is called *topological self-consistency* in [22]. Secondly, for each  $n$ , we compute the Conley index of  $\text{Inv}(B_n)$ , the maximal invariant set of  $B_n$  for (14). The general Conley index theory and an additional assumption (finiteness of the number of expanding direction of vector field) guarantees the construction of a sequence

$$\{z_n \mid z_n \text{ is an equilibrium of (14)}\}_{n \geq m}.$$

Obviously, each  $z_n$  is in  $Z$ , the compact subset of  $H$  defined by the self-consistent bound  $(W, \{a_k^\pm\})$ . Finally we can prove that there is a limit point  $z_*$  of  $\{z_n\}$  by Ascoli-Arzelà's Theorem and  $z_*$  is an equilibrium of the original system (12).

This method was used in [13] for the rigorous continuation of bifurcation branch. We can validate the numerically bifurcated branch away from bifurcation points.

## (2) Global dynamics [4]

Day et.al. applied the previous idea to prove the semi-conjugacy of attractors in infinite dimensional dynamics (12) with gradient-like property to simple finite dimensional attractors. Their method is based on the construction of an attracting compact set  $Z$  in a Hilbert space and the general Conley index theory.

More precisely, we can prove the semi-conjugate theorem as follows. First we compute several sets which contain just one equilibria by Yamamoto's method ([20]) for a fixed parameter value. Secondly we construct a set  $Z$  defined by (13) such that  $Z$  contains all sets we computed as above and that  $Z$  is attracting compact set with all properties of topologically self-consistent a priori bound for the gradient system (12). Thirdly we check the unique- or non-existence of equilibria in small subsets of  $Z$  and count the number of equilibria in  $Z$ . Finally, we study the global structure of  $\text{Inv}(Z)$  by the Conley index and the connection matrix. The computation of the Conley index of each equilibrium and  $\text{Inv}(Z)$  basically follow from [22].

In the next section, we apply such an approach to study the global dynamics of a gradient system at certain parameter values.

## (3) Radii polynomials [2], [5], [10]

Day et.al. proposed another approach to verify the unique existence of equilibria for PDEs with polynomial nonlinearity in [5]. We consider the vector field which is dominated by

$$\dot{u}_k = f_k(u, \nu) := \mu_k u_k + \sum_{p=0}^d \sum_{\sum n_i=k} (c_p)_{n_0} u_{n_1} \dots u_{n_p}, \quad k = 0, 1, \dots, \quad (15)$$

in a separable Hilbert space  $H$ , where  $\mu_k = \mu_k(\nu)$  are the parameter dependent eigenvalues of  $L(\cdot, \nu)$  and  $\{u_n\}$  and  $\{(c_p)_n\}$  are the coefficients of the corresponding expansions of the functions  $u \in H$  and  $c_p(\nu)$  respectively with  $u_n = u_{-n}$  and  $(c_p)_n = (c_p)_{-n}$  for all  $n$ . Let  $m$  be a fixed projection dimension and  $\bar{u} := (u_F, 0) \in H$ , where  $u_F$  is an equilibrium of the following truncation of (15):

$$\dot{u}_k = f_k^{(m)}(u, \nu) := \mu_k u_k + \sum_{p=0}^d \sum_{\sum n_i=k} (c_p)_{n_0} u_{n_1} \dots u_{n_p}, \quad k = 0, 1, m-1.$$

Their approach of the validated computation is to construct an operator  $T$  whose fixed points correspond to equilibria of (15) and show that  $T$  contracts a set of the form

$$W_{\bar{u}}(r) = \bar{u} + \prod_{k=0}^{m-1} [-r, r] \times \prod_{k=m}^{\infty} \left[ -\frac{A_s}{k^s}, \frac{A_s}{k^s} \right] \subset H = P_m H \times (I - P_m) H, \quad (16)$$

where  $r > 0$ ,  $A_s > 0$  are constants and  $s \geq 2$  is the decay rate. In order to verify that  $T$  is a contraction on a set  $W_{\bar{u}}$ , we will have to verify a finite number of polynomial inequality with respect to the radius  $r$ , given by *radii polynomials*. The uniqueness of equilibrium in  $W_{\bar{u}}$  easily follows from the contraction mapping principle for  $T$ .

van den Berg and Lessard applied this approach to the verification of chaos for the Swift-Hohenberg ODE, that is, the existence of an infinite number of equilibria for the Swift-Hohenberg PDE ([2]). Gameiro et.al. applied this approach to the validated computation of equilibria for the Cahn-Hilliard PDE and the Swift-Hohenberg PDE over a large scale of parameter values ([10]).

In the next section, we apply the above method to find a rigorous equilibrium which is unique in a set which forms  $W_{\bar{u}}(r)$  for some  $r > 0$ . Notice that a set  $W_{\bar{u}}(r)$  defines a self-consistent a priori bound for (15) for an appropriate decay rate  $s \geq 0$ . Therefore we can compute the Conley index of an equilibrium in the original system together with constructing an isolating block in  $W_{\bar{u}}(r)$ .



## 5 Application : The Swift-Hohenberg equation

As an application of our approach, we consider the Swift-Hohenberg equation on a finite interval  $I = [0, \ell]$ , where  $\ell = 2\pi/L$  and  $L > 0$ :

$$u_t = E(\nu, u) = \{\nu - (1 + \Delta)^2\} u - u^3, \quad u(\cdot, t) \in L^2(I), \quad (17)$$

with the periodic boundary condition

$$u(x, t) = u(x + \ell, t)$$

and the even function condition

$$u(-x, t) = u(x, t),$$

where  $E : (0, \infty) \times L^2(I) \rightarrow L^2(I)$  and  $\Delta = \partial^2/\partial x^2$ . We remark that  $L^2(I)$  has an orthonormal basis  $\{e^{ikLx} \mid k \in \mathbb{Z}\}$ . By using this basis,  $u(x, t)$  is expressed as follows:

$$u(x, t) = \sum_{k=-\infty}^{\infty} a_k(t) e^{ikLx}, \quad \text{where } (x, t) \in I \times [0, \infty).$$

Since we consider a solution with an even function condition, it is sufficient to consider the following expansion using the Fourier cosine series:

$$u(x, t) = \sum_{k=-\infty}^{\infty} a_k(t) \cos(kLx)$$

with  $a_{-k} = a_k$ .

Hence we assume that  $\{\cos(kLx) \mid k = 0, 1, 2, \dots\}$  is the orthonormal basis for  $L^2(I)$ .

Note that on sets of the form  $W = \prod_k [a_k^-, a_k^+]$  with sufficient (at least quadratic) decay, (17) is equivalent to the following countable system of differential equations

$$\dot{a} = f(a),$$

where  $a = \{a_0, a_1, a_2, \dots\}$  and for  $k = 0, 1, \dots$ ,

$$\dot{a}_k = f_k(a) = \mu_k a_k - \sum_{n_1+n_2+n_3=k, n_i \in \mathbb{Z}} a_{n_1} a_{n_2} a_{n_3}, \quad (18)$$

$$\mu_k := \nu - (1 - k^2 L^2)^2, \quad k = 0, 1, 2, \dots,$$

in  $W$ , which is defined by a self-consistent bound, where  $\{a_k\}$  is a coordinate of  $L^2(I)$  with an orthonormal basis  $\{\cos(kLx)\}$ .

We also remark the Swift-Hohenberg equation has a Lyapunov function:

$$\mathcal{L}(u) = \int_0^\ell \left[ \frac{1}{4} u^4 - \frac{\nu}{2} u^2 + \frac{1}{2} \left( (1 + \Delta)u \right)^2 \right] dx.$$

The existence of a Lyapunov function guarantees that the dynamics generated by (17) is a gradient system.

Here we let  $S_0, S_1$  be  $\mathbb{Z}_2$ -symmetries of an element  $\{a_k\}_{k \geq 0} \in L^2(I)$  as follows:

$$\begin{aligned} S_0 &: a_k \mapsto -a_k, \\ S_1 &: a_{2k} \mapsto a_{2k}, \quad a_{2k+1} \mapsto -a_{2k+1}, \quad k = 0, 1, 2, \dots \end{aligned}$$

In Figure 1, we can expect that a ‘‘pitchfork’’ bifurcation may occur at  $\nu$  close to 0.4761 and 0.62167 (we do not consider other bifurcation points here). In our application, we would like to consider bifurcations as *the change of gradient-like structure of dynamics*. Therefore we study the global dynamics before and after bifurcations and do not use well-known analytic methods such as Lyapunov-Schmidt reduction, which is one of analytic methods to consider the change of local structure of dynamics.

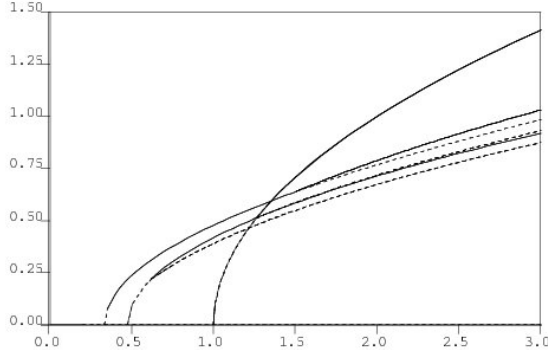


Figure 1: Bifurcation diagram at  $L = 0.65$ .

The horizontal axis is  $\nu$ -coordinate and the vertical axis is  $\|u\|_2$ -coordinate. The solid and the dotted lines indicate stable and unstable equilibria, respectively.

With this mind, we study bifurcations of the Swift-Hohenberg PDE in the following way. First, we find rigorous stationary solutions of (18) at a certain parameter value near a bifurcation point. Secondly, we construct a set  $J_1$  which satisfies conditions (CPF2)-(CPF5). Finally, we construct a C-pitchfork type isolating neighborhood given by

$$[\nu_-, \nu_+] \times J_1.$$

To do so, we would like to obtain a rigorous solution  $a$  of (18). At first, we have to consider numerical solutions  $\tilde{a}$  with  $f(\tilde{a}) \approx 0$ . That is, the zero of the following equation:

$$\dot{a}_k = f_k^{(m)}(a_F) = \mu_k a_k - \sum_{n_1+n_2+n_3=k, |n_i|<m} a_{n_1} a_{n_2} a_{n_3}, \quad k = 0, 1, \dots, m-1. \quad (19)$$

## 5.1 The pitchfork bifurcation from trivial solution

Here we consider the pitchfork bifurcation from a trivial solution in the Swift-Hohenberg equation. Figure.1 shows the bifurcation diagram of SH-equation at  $L = 0.65$ . In Figure.1, it is considered that pitchfork bifurcations from the trivial solution occur at  $\nu \approx 0.334, 0.476$  and  $1$ . Now we let  $\nu_- := 0.47607$ ,  $\nu_+ := 0.47617$  and  $L = 0.65$ .

### 5.1.1 The unique existence of equilibria

First we find rigorous stationary solutions near the numerically obtained bifurcation point near  $\nu \approx 0.4761$ . We can obtain the following result by using radii polynomials which we have mentioned in Section 4.

**Computer Assisted Result (A).** Let  $u_i^{\nu_+}$  and  $r_i^{\nu_+}$ ,  $i = 1^\pm, 2$ , be numerical equilibria defined by the values in Table 1 and let  $s = 4$  and  $A_s = 1$ . Then, for each  $i$ , there exists the unique rigorous equilibrium  $M_i^{\nu_+}$  for the system (18) which is equivalent to (17) in  $W_{u_i^{\nu_+}}(r_i^{\nu_+})$ .

### 5.1.2 The Conley index and the dynamics of invariant sets

Now we consider the Conley indices of stationary solutions. First we construct a local flow which forms

$$Z = \prod_{k=0}^{m-1} [a_k^-, a_k^+] \times \prod_{k \geq m} \left[ -\frac{A_s}{k^s}, \frac{A_s}{k^s} \right], \quad (20)$$

where  $a_k^\pm \in \mathbb{R}$  for  $k = 0, \dots, m-1$ ,  $s \geq 2$  is a decay rate and  $A_s > 0$  is a constant, so that the Conley index theory is applied in  $Z$ . We remark that isolating neighborhoods which form

$$W \times \prod_{k \geq m} \left[ -\frac{A_s}{k^s}, \frac{A_s}{k^s} \right] \subset Z,$$

where  $m$ ,  $A_s$  and  $s$  are the same values as (20) and  $W$  is a subset of  $\mathbb{R}^m$ , make sense *in the topology of a local flow*  $Z$ . Once we construct an isolating block  $B \subset W$  for (19), we can easily compute the Conley index of  $\text{Inv}(B \times \prod_{k \geq m} [-A_s/k^s, A_s/k^s])$  for the Swift-Hohenberg semiflow (17) since all the tail terms are contracting by the construction of a local flow. See Theorem 4.10 in [4] for more details. In this subsection, we set  $m = 10$ ,  $s = 4$  and  $A_s = 1$  as Computer Assisted Result (A).

The idea behind the computation of the Conley index is the construction of isolating block in the  $b$ -coordinate, which is the coordinate obtained by the diagonalization of the linearized Swift-Hohenberg operator at a numerical zero. If we obtain an isolating block in  $b$ -coordinate, we transform its isolating block to the original coordinate. Finally, *if the block is in a validation block (forming  $W_{\bar{u}}(r)$ ) and the Conley index of its isolated invariant set is not trivial*, this isolating block contains the unique stationary solution as the isolated invariant set. Thus, we can obtain the Conley indices of the rigorous stationary solutions.

Now we let  $J_0$  and  $J_1$  which form  $\prod_{k=0}^m [a_k^-, a_k^+] \times \prod_{k > m} [-\frac{A}{k^s}, \frac{A}{k^s}]$  be blocks which contain all the sets  $W_u(r)$  we have obtained in Computer Assisted Result (A). We can prove the following theorem by studying the vector field on its boundary and using the notion of a self-consistent a priori bound.

**Computer Assisted Result (B).** The block  $J_0$  defined by the values in Table 2 is a local flow for  $\nu \in [0.47607, 0.47617]$ . Next let  $(L, \nu) = (0.65, 0.47617)$ . The Conley index of  $M_i^{\nu+}$ ,  $i = 1^\pm$ , is as follows:

$$CH_n(\{M_i^{\nu+}\}, \varphi^{\nu+}) \cong \begin{cases} \mathbb{Z}_2 & n = 1 \\ 0 & n \neq 1 \end{cases}.$$

The Conley index of  $M_2^{\nu+}$  is

$$CH_n(\{M_2^{\nu+}\}, \varphi^{\nu+}) \cong \begin{cases} \mathbb{Z}_2 & n = 2 \\ 0 & n \neq 2 \end{cases}.$$

**Computer Assisted Result (C).** The block  $J_1$  defined by the values in Table 3 is an isolating block in  $J_0$  for  $\nu \in [\nu_-, \nu_+]$  and the Conley index of  $\text{Inv}(J_1)$  is

$$CH_n(\text{Inv}(J_1), \varphi^{\nu+}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

The following lemma shows the  $\mathbb{Z}_2$ -symmetry of invariant sets.

**Lemma 5.1.** *Let  $N_0$  and  $N_1$  be isolating neighborhoods. If  $gN_i = N_{1-i}$ ,  $i = 0, 1$ , for  $g \in \mathbb{Z}_2$ , then  $g\text{Inv}(N_i) = \text{Inv}(N_{1-i})$ ,  $i = 0, 1$ . In particular, if  $N$  is an isolating neighborhood with  $gN = N$ , then  $g\text{Inv}(N) = \text{Inv}(N)$ .*

Now we consider the dynamics of  $\text{Inv}(J_1)$ . To this end, we have to check the number of equilibria in  $\text{Inv}(J_1)$ . We combine the following non-existence result with the unique-existence. We set

$$B = B_F \times B_I, \text{ where } B_F = \prod_{k=0}^{m-1} [a_k^-, a_k^+] \text{ and } B_I = \prod_{k=m}^{\infty} [a_k^-, a_k^+],$$

We define the interval vector  $v_k(a_F)$  for each  $a_F = (a_0, \dots, a_{m-1}) \in B_F$  as follows:

$$v_k(a_F) = \begin{cases} \tilde{a}_k - a_k & \text{for } |k| \leq m, \\ 0 & \text{for } |k| > m. \end{cases}$$

**Theorem 5.2** ([4]). *If there exist a point  $a_F^* \in B_F$  and  $k \in \mathbb{N}$  such that*

$$0 \notin f_k(a_F^*, B_I) + (Df_k(B) \cdot v(a_F^*))$$

*for the vector field, then  $B = B_F \times B_I$  contains no stationary solutions in (18).*

We obtain the following result counting the number of fixed points.

**Theorem 5.3.** *The block  $J_1$  as Table 3 contains exactly three equilibria,  $M_i^{\nu+}$ ,  $i = 1^{\pm}, 2$ , at  $\nu_+$ , and exactly one equilibrium, 0, at  $\nu_-$ . Moreover, the collection of equilibria*

$$\{M_{1+}^{\nu+}, M_{1-}^{\nu+}, M_2^{\nu+}\}$$

*is a Morse decomposition of  $\text{Inv}(J_1)$  at  $\nu_+$  with the admissible ordering  $1^+ < 2$ ,  $1^- < 2$ .*

We refer to the similar result, in detail, in sub-subsection 5.2.5. Now we have checked that  $[0.47607, 0.47617] \times J_1$  satisfies the conditions (CPF1)-(CPF5). Moreover, we know that  $[0.47607, 0.47617] \times J_1$  is an isolating neighborhood (in the local flow  $[0.47607, 0.47617] \times J_1$  itself) by calculations of the value of the vector field on  $[0.47607, 0.47617] \times \partial J_1$ .

Then, we have proved the Theorem 5.4.

**Theorem 5.4.** *If  $L = 0.65$ , then the compact subset in  $\mathbb{R} \times L^2(0, 2\pi/L)$  given by*

$$[0.47607, 0.47617] \times J_1$$

*is a C-pitchfork type isolating neighborhood over  $[0.47607, 0.47617]$  with type-S admissible ordering, where  $J_1 = \prod_{k \geq 0} [a_k^-, a_k^+]$  is given by Table 3. Therefore a C-pitchfork bifurcation occurs at  $\nu_{PF} \in 0.47612 + [-5.0 \times 10^{-5}, 5.0 \times 10^{-5}]$ .*

## 5.2 The secondary pitchfork bifurcation

In this subsection, we consider the bifurcation from a nontrivial solution. The second branch (bifurcated from the trivial solution at  $\nu \approx 0.476$ ) in Figure.1 numerically suggests that the SH-equation undergoes a secondary bifurcation at  $\nu \approx 0.62167$  (this value is obtained by AUTO ([6])).

Here we construct an isolating neighborhood of C-pitchfork type by the following approach. First, we construct a local flow  $J_2$  over a certain interval  $[\nu_-, \nu_+]$  which contains stationary solutions before and after a bifurcation. Of course, this region may contain stationary solutions which continue over  $[\nu_-, \nu_+]$ . Then we try to remove smaller isolating blocks each of which contains a single stationary solution.

Notice that for two isolating neighborhoods  $N_1$  and  $N_2$  with  $N_1 \supset N_2$ , the difference  $N_1 \setminus N_2$  may not be isolating in general. However, we can prove that the set  $N_1 \setminus N_2$  is isolating if  $N_1$  is isolating and if  $N_2$  is also isolating such that  $\text{Inv}(N_2)$  consists of just one chain recurrent component. Using this property, we can construct an isolating neighborhood from the local flow. Finally, we check if the isolating neighborhood in  $J_2$  is of C-pitchfork type.

Now we let  $\nu_- := 0.62163$ ,  $\nu_+ := 0.62173$  and  $L := 0.65$ .

### 5.2.1 The unique existence of equilibria

First we find rigorous equilibria close to a bifurcation point. As Computer Assisted Result (A), we can obtain the following result by using radii polynomials which we have mentioned in Section 4.

**Computer Assisted Result (D).**

(1) Let  $u_i^{\nu_-}, r_i^{\nu_-}, i \in \{0_+, 0_-, 1_+, 1_-, 2\}$ , be numerical equilibria defined by values in Table 4,  $s = 6$  and  $A_s = 1.0$ . Then each set  $W_{u_i^{\nu_-}}(r_i^{\nu_-})$  contains the unique rigorous equilibrium  $M_i^{\nu_-}$  of the equation (17) at  $\nu = \nu_-$ .

(2) Let  $u_i^{\nu_+}$  and  $r_i^{\nu_+}, i \in \{0_+, 0_-, 1_+, 1_{++}, 1_{+-}, 1_-, 1_{-+}, 1_{--}, 2\}$ , be numerical equilibria defined by values in Table 5,  $s = 6$  and  $A_s = 1.0$ . Then each set  $W_{u_i^{\nu_+}}(r_i^{\nu_+})$  contains the unique equilibrium  $M_i^{\nu_+}$  of the equation (17) at  $\nu = \nu_+$ , respectively.

### 5.2.2 Construction of a local flow

Now we construct a local flow. First, let  $J = \prod_{k \geq 0} [a_k^-, a_k^+]$  be the smallest block which contains all the sets  $W_u(r)$  we have obtained in Computer Assisted Result (D). Second, we compute the vector field on the boundary of  $J$ . That is,

$$f_k(a)|_{a_k=a_k^+} \text{ and } f_k(a)|_{a_k=a_k^-}, a \in J,$$

for each  $k$  by interval arithmetic. If, for  $\nu \in [\nu_-, \nu_+]$ , all  $k$  are contracting directions, namely,

$$\begin{aligned} f_k(a) &< 0 \text{ for all } a \in J \text{ with } a_k = a_k^+, \\ f_k(a) &> 0 \text{ for all } a \in J \text{ with } a_k = a_k^-, \end{aligned}$$

then  $J$  is a local flow of  $\varphi^\nu$  for  $\nu \in [\nu_-, \nu_+]$ . If the condition is not satisfied, we change the size of  $J$  so that  $J$  satisfies the condition as above. By using this we obtain a local flow.

**Computer Assisted Result (E).** Let

$$J_2 = \prod_{0 \leq k \leq 9} [a_k^-, a_k^+] \times \prod_{k \geq 10} \left[-\frac{1.0}{k^6}, \frac{1.0}{k^6}\right]$$

be the set defined by the values in Table 6. Then the set  $J_2$  is a local flow for  $\nu \in [\nu_-, \nu_+]$ . The Conley index of  $\text{Inv}(J_2)$  is

$$CH_n(\text{Inv}(J_2), \varphi^\nu) \cong \begin{cases} \mathbb{Z}_2 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

for all  $\nu \in [\nu_-, \nu_+]$ .

Throughout the rest of this paper, we consider the dynamics on the local flow  $J_2$ . Since  $J_2$  is compact in  $L^2(I)$ , we can apply the Conley index theory to *the semiflow on  $J_2$* . We remark that  $J_2$  is an isolating neighborhood in  $J_2$  itself because all directions are contracting.

### 5.2.3 Conley indices of equilibria

Verifying the unique existence of steady-states together with non-existence of them, we know the number of steady-states in  $J_2$  at  $\nu_\pm$ .

**Computer Assisted Result (F).** At  $\nu = \nu_-$ , the set  $J_2 \subset L^2(0, 2\pi/0.65)$  contains exactly five equilibria,  $M_i^{\nu_-}, i = 0^\pm, 1^\pm, 2$ . Similarly, at  $\nu = \nu_+$ , the set  $J_2$  contains exactly nine equilibria,  $M_j^{\nu_+}, j = 0^\pm, 1^\pm, 1^{\pm\pm}, 2$ .

The following result which is obtained by using the notion of a self-consistent a priori bound shows the Conley index of each equilibrium.

**Computer Assisted Result (G).** The Conley index of each equilibrium in Computer Assisted Result (F) is defined by groups in Table 7.

### 5.2.4 Removing isolating subneighborhoods

Now we construct an isolating neighborhood of C-pitchfork type. See Figure.1 again. We can expect three steady-states  $M_{0\pm}^\nu$  (equilibria on branches which are bifurcated from the trivial equilibrium at  $\nu \approx 0.334$ ) and  $M_2^\nu$  (trivial equilibrium) continue over  $[\nu_-, \nu_+] = [0.62163, 0.62173]$ . Therefore we try to remove isolating blocks which contain such equilibria.

First, we remove  $M_{0\pm}^\nu$ .

**Proposition 5.5.** *Two equilibria  $M_{0\pm}^\nu$  continue over  $[\nu_-, \nu_+]$  with certain isolating neighborhoods  $N_{0\pm}^\nu$  which satisfies*

$$CH_n(\text{Inv}(N_{0+}^\nu)) \cong CH_n(\text{Inv}(N_{0-}^\nu)) \cong \begin{cases} \mathbb{Z}_2 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

for each  $\nu \in [\nu_-, \nu_+]$ .

We can construct  $N_{0\pm}^\nu$  in Proposition 5.5 by the method in [22]. In particular, these sets are attracting and hence  $\text{Inv}(N_{0\pm}^\nu)$  are attractors in  $\text{Inv}(J_2)$  for  $\nu \in [\nu_-, \nu_+]$ . Therefore the set

$$R_1^\nu := \text{Inv}(J_2 \setminus (\text{Inv}(N_{0+}^\nu) \cup \text{Inv}(N_{0-}^\nu)))$$

is the dual repeller of  $\text{Inv}(N_{0+}^\nu) \cup \text{Inv}(N_{0-}^\nu)$  in  $\text{Inv}(J_2)$  for each  $\nu \in [\nu_-, \nu_+]$ . There exists an isolating neighborhood of  $R_1^\nu$ . We let this neighborhood be  $N_{R_1^\nu}$ .

Next, we remove  $M_2^\nu$  from  $N_{R_1^\nu}$ .

**Proposition 5.6.** *The equilibrium  $M_2^\nu$  continues over  $[\nu_-, \nu_+]$ .*

We remark that the trivial solution is isolated over  $[\nu_-, \nu_+]$  and hence this is a chain recurrent component of  $\text{Inv}(N_{R_1^\nu})$  because the Swift-Hohenberg flow is gradient-like. Now we can prove the following lemma.

**Lemma 5.7.** *Let  $N$  be an isolating neighborhood in a locally compact metric space  $X$  for the flow  $\varphi$  and  $S$  be an isolated invariant set which consists of just one chain recurrent component of  $\text{Inv}(N)$ . Then there exists an isolating neighborhood  $\tilde{N}$  of  $S$  such that  $\text{Inv}(N \setminus \tilde{N})$  is also isolated.*

**Remark 5.8.** The chain recurrent set  $\mathcal{R}(\varphi)$  of a flow  $\varphi$  on a metric space  $M$  is defined by

$$\mathcal{R}(\varphi) := \{x \in M \mid \text{there is an } \epsilon\text{-chain from } x \text{ to } x \text{ of length greater than } T \text{ for all } \epsilon > 0 \text{ and for all } T > 1\}.$$

There is an equivalence relation on  $\mathcal{R}(\varphi)$ . Namely, a relation  $x \sim y$  on  $\mathcal{R}(\varphi)$  is defined if for any  $\epsilon > 0$ , there exists an  $\epsilon$ -chain from  $x$  to  $y$  and an  $\epsilon$ -chain from  $y$  to  $x$ . An equivalent class is called a *chain recurrent component* of  $\mathcal{R}(\varphi)$ .

For the proof of this lemma, we use *Conley's Fundamental Theorem of Dynamics*. Namely, a flow  $\varphi$  on a compact metric space has a Lyapunov function  $L$  which is strictly decreasing off the chain recurrent set  $\mathcal{R}(\varphi)$  of  $\varphi$  and such that  $L(\mathcal{R}(\varphi))$  is a nowhere dense subset of  $\mathbb{R}$ .

*Proof.* By Conley's Fundamental Theorem of Dynamics, there exists a function  $L : \text{Inv}(N) \rightarrow [0, 1]$  such that  $L$  is strictly decreasing off the chain recurrent set of  $\text{Inv}(N)$  with respect to time  $t$  on trajectories. Moreover,  $L$  is constant on each chain recurrent component of  $\text{Inv}(N)$  and this acquires different values between different components and the image of the chain recurrent set is nowhere dense in  $[0, 1]$ . We let  $R(N, \varphi)$  the chain recurrent set in  $\text{Inv}(N, \varphi)$ . Since  $S$  consists of just one chain recurrent component, then there exists  $c \in [0, 1]$  such that  $S \subset L^{-1}(c)$ . Since the image of the chain recurrent set is nowhere dense in  $[0, 1]$  and  $S$  is isolated, then there exists  $\epsilon > 0$  such that the chain recurrent component in  $L^{-1}(c - \epsilon, c + \epsilon)$  is only  $S$  and that  $(L^{-1}(\{c - \epsilon\}) \cup L^{-1}(\{c + \epsilon\})) \cap R(N, \varphi) = \emptyset$ . Thus  $N_{c,\epsilon} := L^{-1}([c - \epsilon, c + \epsilon])$  is an isolating neighborhood which isolates  $S$ .

Next, we extend  $L$  to  $N$  by Tietze's extension theorem. We write such extended function  $L$  again. We take an open small ball  $B(L^{-1}(c + \mu) \cap \text{Inv}(N), \delta_\mu)$  for each  $\mu \in [-\epsilon, \epsilon]$ ,  $\delta_\mu > 0$  so that  $B(L^{-1}(c + \mu) \cap \text{Inv}(N), \delta_\mu) \subset \text{int}(N)$  for  $\mu \in [-\epsilon, \epsilon]$ . Then a collection of open balls

$$\{B(L^{-1}(c + \mu) \cap \text{Inv}(N), \delta_\mu)\}_{\mu \in [-\epsilon, \epsilon]}$$

is an open cover of  $L^{-1}([c-\epsilon, c+\epsilon]) \cap \text{Inv}(N)$ . Since  $[c-\epsilon, c+\epsilon]$  is compact, there exist  $\mu_1, \dots, \mu_n \in [-\epsilon, \epsilon]$  such that  $\{B(L^{-1}(c + \mu_i) \cap \text{Inv}(N), \delta_{\mu_i})\}_{i=1, \dots, n}$  is an open subcovering of  $L^{-1}([c-\epsilon, c+\epsilon]) \cap \text{Inv}(N)$ . Now we define a set  $B$  as

$$B := \left( \bigcup_{\mu=\mu_1, \dots, \mu_n, \pm\epsilon} \overline{B(L^{-1}(c + \mu) \cap \text{Inv}(N), \delta_{\mu})} \right) \cap N_{c, \epsilon}.$$

For any  $\mu \in (-\epsilon, \epsilon)$ ,  $\text{Inv}(N) \cap (\partial B \cap L^{-1}(\{c + \mu\})) = \emptyset$ . For  $\mu = \pm\epsilon$ ,  $\text{Inv}(N) \cap (\partial B \cap L^{-1}(\{c \pm \epsilon\})) \subset \text{int}_{L^{-1}(\{c \pm \epsilon\})}(\partial B \cap L^{-1}(\{c \pm \epsilon\}))$ , where  $\text{int}_A(U)$  is the interior of  $U$  in the topology of  $A$ . Since for any  $x \in \text{Inv}(N) \cap L^{-1}(\{c \pm \epsilon\})$ , there exists  $\epsilon_x > 0$  such that

$$x \cdot (0, \epsilon_x) \cap B = \emptyset \text{ or } x \cdot (-\epsilon_x, 0) \cap B = \emptyset,$$

then both  $B$  and  $\overline{N \setminus B}$  are isolating. We can also prove our claim even if  $c$  is equal to 0 or 1 by similar arguments.  $\square$

Since  $M_2^\nu$  is isolated and continues over  $[\nu_-, \nu_+]$ , then there exists an isolating neighborhood  $N_2^\nu$  of  $M_2^\nu$  such that  $\text{Inv}(N \setminus N_2^\nu)$  is also isolated for  $\nu \in [\nu_-, \nu_+]$ . By the robustness of isolating neighborhoods, the set

$$V_{\nu_0} := \{\nu \mid N_2^{\nu_0} \text{ is an isolating neighborhood of } M_2^\nu \text{ and } \overline{N \setminus N_2^{\nu_0}} \text{ is also isolating for } \varphi^\nu\}$$

is open in  $[\nu_-, \nu_+]$ . Thus the collection of such open sets

$$\mathcal{V} := \{V_\nu\}_{\nu \in [\nu_-, \nu_+]}$$

is an open covering of  $[\nu_-, \nu_+]$ . Therefore there exists finite number of parameter values  $\nu_1, \dots, \nu_n \in [\nu_-, \nu_+]$  such that the subcollection  $\{V_{\nu_i}\}_{i=1, \dots, n}$  covers  $[\nu_-, \nu_+]$ . Thus if we define the set  $N_2 \subset [\nu_-, \nu_+] \times L^2(I)$  satisfying as follows:

$$N_2^\nu = N_2|_{\{\nu\} \times L^2(I)} := N_2^{\nu_i} \text{ for some } i \text{ for each } \nu \in [\nu_-, \nu_+],$$

then  $N_2$  is an isolating neighborhood which  $N_2^\nu$  isolates  $M_2^\nu$  for  $\nu \in [\nu_-, \nu_+]$  and which  $(\bigoplus_{\nu \in [\nu_-, \nu_+]} N_{R_1^\nu}) \setminus N_2$  is isolating. Then we have obtained the desired isolating neighborhood.

We define  $N_{\text{PF}} := (\bigoplus_{\nu \in [\nu_-, \nu_+]} N_{R_1^\nu}) \setminus N_2$ .

### 5.2.5 The final result

We check if  $N_{\text{PF}}$  is an isolating neighborhood of C-pitchfork type. First, by our construction,  $\text{Inv}(N_{\text{PF}}^{\nu_-})$  consists of exactly two equilibria,  $M_{1\pm}^{\nu_-}$ . They are  $\mathbb{Z}_2$ -symmetric with respect to the symmetry  $S_1$ . Next,  $\text{Inv}(N_{\text{PF}}^{\nu_+})$  consists of exactly six equilibria,

$$M_j^{\nu_+}, \quad j = +, -, ++, +-, -+, --,$$

and connecting orbits between them. Now we would like to know the dynamics of  $\text{Inv}(N_{\text{PF}}^{\nu_+})$  for the sake of completeness. We know it by using our above verifications and the connection matrix. To this end, we have to know the Conley index of  $\text{Inv}(N_{\text{PF}}^{\nu_+})$ .

We show the following lemma to know the Conley index of  $\text{Inv}(N_{\text{PF}}^{\nu_+})$ .

**Lemma 5.9.** *Consider the isolated invariant set  $\text{Inv}(J_2, \varphi^{\nu_-})$  with Morse decomposition*

$$\mathcal{M}(\text{Inv}(J_2, \varphi^{\nu_-})) := \{M_i^{\nu_-} \mid i = 0^\pm, 1^\pm, 2\}.$$

*An admissible partial order on the indexing set  $\{0^\pm, 1^\pm, 2\}$  is  $2 > 1^\pm > 0^\pm$ .*

*Proof.* The fact that  $\mathcal{M}(\text{Inv}(J_2, \varphi^{\nu_-}))$  is a Morse decomposition of  $\text{Inv}(J_2, \varphi^{\nu_-})$  follows from Computer assisted Result (F) and the existence of Lyapunov function  $\mathcal{L}$ . By the  $\mathbb{Z}_2$ -symmetry of the Swift-Hohenberg flow,  $\mathcal{L}(M_{0-}^{\nu_-}) = \mathcal{L}(-M_{0+}^{\nu_-})$  and  $\mathcal{L}(M_{1-}^{\nu_-}) = \mathcal{L}(-M_{1+}^{\nu_-})$ , which implies that there cannot be any connecting orbits between  $M_{0+}^{\nu_-}$  and  $M_{0-}^{\nu_-}$ ,  $M_{1+}^{\nu_-}$  and  $M_{1-}^{\nu_-}$ . Therefore we can choose an admissible

ordering for which  $0^\pm, 1^\pm$  are unrelated, respectively. To determine the ordering we turn to the connection matrix. Let  $\Delta$  be a connection matrix defined on

$$CH_*(M_{0+}^{\nu-}) \oplus CH_*(M_{0-}^{\nu-}) \oplus CH_*(M_{1+}^{\nu-}) \oplus CH_*(M_{1-}^{\nu-}) \oplus CH_*(M_2^{\nu-}).$$

Since  $\Delta$  is upper triangular and  $\text{Inv}(J_2) \neq 0$ , then  $\Delta$  must take the form

$$\Delta = \begin{bmatrix} 0 & 0 & \alpha & \beta & 0 \\ 0 & 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the condition of  $\Delta$ , (CM2) and (CM3), we can conclude that  $(\eta, \mu) \neq (0, 0)$  and  $(\alpha, \beta, \gamma, \delta) \neq (0, 0, 0, 0)$ . This implies that we can choose  $2 > 1^\pm > 0^\pm$  as an admissible ordering.  $\square$

**Remark 5.10.** By the discussion in [4], the connection matrix takes the following form:

$$\Delta = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Proposition 5.11.** *The Conley index of  $\text{Inv}(N_{\text{PF}}^{\nu+})$  is as follows:*

$$CH_n(\text{Inv}(N_{\text{PF}}^{\nu+})) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}.$$

*Proof.* It is sufficient to know the Conley index of  $\text{Inv}(N_{\text{PF}}^{\nu-})$  by the continuation property of the Conley index.

First, we compute the Conley index of  $N_{R_1}^{\nu-}$ , which is the isolating neighborhood removed  $N_{0^\pm}^{\nu-}$  from  $J_2$ . By Lemma 5.9,  $M_{0^\pm}^{\nu-} = \text{Inv}(N_{0^\pm}^{\nu-})$  are attractors in  $\text{Inv}(J_2, \varphi^{\nu-})$  for the Morse decomposition  $\mathcal{M}(\text{Inv}(J_2, \varphi^{\nu-}))$  as in Lemma 5.9 with an admissible ordering  $2 > 1^\pm > 0^\pm$ , there exists the following A-R pair long exact sequence,

$$\begin{aligned} \cdots \rightarrow CH_n(\text{Inv}(N_{1+}^{\nu-})) \rightarrow CH_n(\text{Inv}(J_2, \varphi^{\nu-})) \rightarrow \\ CH_n(\text{Inv}(J_2 \setminus N_{1+}^{\nu-}, \varphi^{\nu-})) \xrightarrow{\partial} CH_{n-1}(\text{Inv}(N_{1+}^{\nu-})) \rightarrow \dots, \end{aligned}$$

and this implies that the Conley index of  $\text{Inv}(J_2^{\nu-} \setminus N_{1+}^{\nu-})$  is trivial. Similarly, there exists the following A-R pair long exact sequence,

$$\begin{aligned} \cdots \rightarrow CH_n(\text{Inv}(N_{1-}^{\nu-})) \rightarrow CH_n(\text{Inv}(J_2 \setminus N_{1+}^{\nu-}, \varphi^{\nu-})) \rightarrow \\ CH_n(\text{Inv}(N_{R_1}^{\nu-})) \xrightarrow{\partial} CH_{n-1}(\text{Inv}(N_{1-}^{\nu-})) \rightarrow \dots, \end{aligned}$$

and this implies that the Conley index of  $\text{Inv}(N_{R_1}^{\nu-})$  is as follows:

$$CH_n(\text{Inv}(N_{R_1}^{\nu-})) \cong \begin{cases} \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, by Lemma 5.9,  $M_2^{\nu-}$  is a repeller in  $N_{R_1}^{\nu-}$  for the Morse decomposition

$$\mathcal{M}(N_{R_1}^{\nu-}) := \{M_i^{\nu-} \mid i = 1^\pm, 2\}$$

with an admissible ordering  $2 > 1^\pm$ , there exists the following A-R pair long exact sequence,



$$\cdots \rightarrow CH_n(\text{Inv}(J_{\text{PF}}^{\nu-})) \rightarrow CH_n(\text{Inv}(N_{R_1}^{\nu-})) \rightarrow CH_n(\text{Inv}(N_2^{\nu-})) \xrightarrow{\partial} CH_{n-1}(\text{Inv}(J_{\text{PF}}^{\nu-})) \rightarrow \cdots,$$

where  $N_2^{\nu-}$  is the isolating neighborhood defined in subsection 5.2.4, it is sufficient to consider the following exact sequence:

$$0 \rightarrow CH_2(\text{Inv}(N_2^{\nu-})) \xrightarrow{\partial} CH_1(\text{Inv}(J_{\text{PF}}^{\nu-})) \rightarrow CH_1(\text{Inv}(N_{R_1}^{\nu-})) \rightarrow 0.$$

Since we assume that the coefficient of homology groups is a field  $\mathbb{Z}_2$  here, all homology groups are free  $\mathbb{Z}_2$ -modules. In particular,  $CH_1(\text{Inv}(N_{R_1}^{\nu-}))$  is  $\mathbb{Z}_2$ -free. Therefore this sequence decomposes. Hence

$$CH_1(\text{Inv}(J_{\text{PF}}^{\nu-})) \cong CH_2(\text{Inv}(N_2^{\nu-})) \oplus CH_1(\text{Inv}(N_{R_1}^{\nu-}))$$

and the Conley index of  $\text{Inv}(N_{\text{PF}}^{\nu-})$  is as follows:

$$CH_n(\text{Inv}(J_{\text{PF}}^{\nu-})) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n=1 \\ 0 & \text{otherwise} \end{cases}.$$

□

Using the above result, we know the dynamics of  $\text{Inv}(N_{\text{PF}}^{\nu+})$ .

**Proposition 5.12** (Dynamics of  $\text{Inv}(N_{\text{PF}}^{\nu+})$ ). *Consider the collection of equilibria*

$$\mathcal{M}(\text{Inv}(N_{\text{PF}}^{\nu+})) := \{M_i^{\nu+} \mid i = 1^\pm, 1^{\pm\pm}\}.$$

*This collection is a Morse decomposition of  $\text{Inv}(N_{\text{PF}}^{\nu+})$ . An admissible ordering on the indexing set  $\{1^\pm, 1^{\pm\pm}\}$  is  $1^{\pm\pm} > 1^\pm$ .*

*Proof.* First, the collection of equilibria

$$\mathcal{M}(\text{Inv}(N_{\text{PF}}^{\nu+})) := \{M_i^{\nu+} \mid i = 1^\pm, 1^{\pm\pm}\}$$

is a Morse decomposition of  $\text{Inv}(N_{\text{PF}}^{\nu+})$  by Computer assisted Result (F) and the existence of the Lyapunov function  $\mathcal{L}$ . By the symmetry of the Swift-Hohenberg,  $\mathcal{L}(M_{1-}^{\nu+}) = \mathcal{L}(-M_{1+}^{\nu+})$ , which implies that there cannot be any connections between  $M_{1+}^{\nu+}$  and  $M_{1-}^{\nu+}$ . Therefore  $M_{1+}^{\nu+}$  and  $M_{1-}^{\nu+}$  are unrelated. Similarly,  $M_{1^{++}}^{\nu+}$ ,  $M_{1^{+-}}^{\nu+}$ ,  $M_{1^{-+}}^{\nu+}$  and  $M_{1^{--}}^{\nu+}$  are unrelated each other. To determine the ordering we turn to the connection matrix. We let  $\Delta$  be a connection matrix defined on

$$CH_*(M_{1+}^{\nu+}) \oplus CH_*(M_{1-}^{\nu+}) \oplus$$

$$CH_*(M_{1^{++}}^{\nu+}) \oplus CH_*(M_{1^{+-}}^{\nu+}) \oplus CH_*(M_{1^{-+}}^{\nu+}) \oplus CH_*(M_{1^{--}}^{\nu+}).$$

Since  $\Delta$  is a boundary operator, then it must take the following form:

$$\Delta = \begin{bmatrix} 0 & 0 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & \mu & \rho & \lambda & \xi \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the  $\mathbb{Z}_2$ -symmetry  $S_0$ , the matrix forms as following:

$$\Delta = \begin{bmatrix} 0 & 0 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & \gamma & \delta & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly by the  $\mathbb{Z}_2$ -symmetry  $S_1$ , the matrix forms as following:

$$\Delta = \begin{bmatrix} 0 & 0 & \alpha & \beta & \beta & \alpha \\ 0 & 0 & \beta & \alpha & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the condition of the connection matrix (CM3), we conclude that  $(\alpha, \beta) \neq (0, 0)$ . This implies that we can choose an admissible ordering,  $1^{\pm\pm} > 1^\pm$ .  $\square$

We let  $M(0_+) := M_{1^{++}}^{\nu_+} \cup M_{1^{-+}}^{\nu_+}$ ,  $M(0_-) := M_{1^{+-}}^{\nu_+} \cup M_{1^{--}}^{\nu_+}$  and  $M(1) := M_{1^+}^{\nu_+} \cup M_{1^-}^{\nu_+}$ . The collection

$$\{M(0_+), M(0_-), M(1)\}$$

is a Morse decomposition of  $\text{Inv}(N_{\text{PF}}^{\nu_+})$  with an admissible ordering  $0_\pm > 1$ . Obviously,  $M(0_+)$  and  $M(0_-)$  map each other by  $S_1$  and  $M(1)$  is  $\mathbb{Z}_2$ -symmetric. All the Conley indices of  $M(0_+)$ ,  $M(0_-)$  and  $M(1)$  are nontrivial.

Summarizing above discussions, we have proved the final result.

**Theorem 5.13.** *The set  $N_{\text{PF}}$  is a C-pitchfork type isolating neighborhood over  $[\nu_-, \nu_+] = [0.62163, 0.62173]$  with type-U admissible ordering in the local flow  $J_2$ . Therefore a C-pitchfork bifurcation occurs at  $\nu_{\text{PF}} \in 0.62168 + [-5.0 \times 10^{-5}, 5.0 \times 10^{-5}]$ .*

**Remark 5.14.** All codes are written in C++ and the interval arithmetic package C-XSC available in <http://www.xsc.de/> and they were run on an Intel Core 2 Duo U7500 (1.02GHz) with 1 GB of memory.

## 6 Conclusion.

We have identified a method to capture bifurcations in terms of C-type bifurcations. We have also applied the method to the rigorous verification of bifurcations in the dynamics generated by the Swift-Hohenberg equation.

P.Zgliczyński has extended the method of him and Mischaikow to rigorously obtain the precise saddle-node and pitchfork bifurcation in [21]. He has estimated the derivatives of higher order of the functions which are defined by a partial differential equation at a stationary solution. We know the occurrence of standard bifurcations by his method.

As we have shown in the end of Section 3, in our approach, we take no notice of precise recurrent structure of dynamics. At a glance, dynamics (on a compact space) consists of recurrent structure and gradient-like structure (by Conley's Fundamental Theorem of Dynamical Systems). Our approach shows that we can easily understand global dynamics and bifurcations only attending its gradient-like structure, although we can obtain only coarse results, because recurrent structure contains periodic, chaotic or other complicated structures in general. Such a view of dynamics is compatible with computations of global multiparameterized dynamics as database [1].

We end the paper by giving remarks and open problems.

- (1) We have discussed only saddle-node and pitchfork bifurcations in this paper. But, we can also consider the transcritical, the Hopf or other bifurcations. The consideration of the methods to capture these bifurcations rigorously by topological methods are themes in future.
- (2) In this paper, we constructed an isolating neighborhood of C-pitchfork type by constructing a local flow and removing isolating neighborhoods containing steady-states which continue over  $[\nu_-, \nu_+]$  and hence it is quite large. This time the author tried to construct isolating neighborhoods containing a bifurcation point from non-trivial equilibrium by the same method to capture a bifurcation point from trivial equilibrium or the method in [22]. However, he has never succeeded to construct them. C-type isolating neighborhoods would have quite complicated shapes if we could obtain them.

How can we construct C-type isolating neighborhoods effectively as small as possible?

There is a hint to construct a C-type isolating neighborhood in the paper by Pilarczyk ([15]). He shows there the algorithm to construct an index pair of an isolated invariant set for finite dimensional dynamical systems by using the Lohner method, which is to compute the bound of rigorous integration of vector field by computing the bound of time- $t$  map. It is enough to use the method in his paper for finite dimensional systems. However, our problems contain infinite dimensional ones and stiff ones. Thus we should extend his method for such systems.

- (3) How can we obtain the estimate of the parameter value where the bifurcation occurs as close as possible? Estimates in this paper depend on verifications of the unique existence of equilibria very much. It is difficult to verify the unique existence of a certain equilibrium close to a bifurcation point because the linearized operator at the equilibrium has an eigenvalue with very small real part. However, it is not originally necessary to verify the unique existence of *all* equilibria and nonexistence to check the condition of C-type neighborhood. Namely, it is enough to know the structure of Morse decompositions. For example, when we want to verify conditions of a neighborhood of C-saddle-node type, especially, (CSN2) and (CSN3), it is enough to discover an isolating neighborhood with trivial index and an attractor, which we can compute the Conley index, in it because we can automatically obtain the dual repeller of the attractor with nontrivial index by the Conley index theory. Thus we can obtain the estimate of parameter values as well as possible if we can even construct Morse decompositions.

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## Tables of numerical results.

Table 1: Equilibria at  $\nu_+ = 0.47617$ .

$k$	$u_{1+}^{\nu_+}$	$u_2^{\nu_+}$
0	$+7.6915641017 \times 10^{-27}$	0.0
2	$+4.8331541002 \times 10^{-3}$	0.0
4	$+6.1621665460 \times 10^{-29}$	0.0
6	$-5.5981478867 \times 10^{-10}$	0.0
8	$+8.5684788884 \times 10^{-34}$	0.0
other $k$	0.0	0.0
$r_i^{\nu_+}$	$2.43151131902 \times 10^{-5}$	$8.9760954800 \times 10^{-6}$

Note that  $u_{1-}^{\nu_+} = S_0 u_{1+}^{\nu_+}$  and that  $r_{1-}^{\nu_+} = r_{1+}^{\nu_+}$ .

Table 2:  $J_0 := \prod_{k \geq 0} [a_k^-, a_k^+] \subset L^2(0, \ell)$  : a local flow for  $\nu \in [0.47607, 0.47617]$ .

$k$	$a_k^-$	$a_k^+$
0	$-2.187274096935 \times 10^{-2}$	$+2.187274096935 \times 10^{-2}$
1	$-2.231206783750635 \times 10^{-1}$	$+2.231206783750635 \times 10^{-1}$
2	$-4.8574508746215 \times 10^{-2}$	$+4.8574508746215 \times 10^{-2}$
3	$-6.8294730330365 \times 10^{-3}$	$+6.8294730330365 \times 10^{-3}$
4	$-3.645456828225 \times 10^{-4}$	$+3.645456828225 \times 10^{-4}$
5	$-7.29091365645 \times 10^{-5}$	$+7.29091365645 \times 10^{-5}$
6	$-4.86072106712 \times 10^{-5}$	$+4.86072106712 \times 10^{-5}$
7	$-2.43030455215 \times 10^{-5}$	$+2.43030455215 \times 10^{-5}$
8	$-2.43030455215 \times 10^{-5}$	$+2.43030455215 \times 10^{-5}$
9	$-2.43030455215 \times 10^{-5}$	$+2.43030455215 \times 10^{-5}$
$k \geq 10$	$-1.0/k^4$	$1.0/k^4$

Table 3:  $J_1 := \prod_{k \geq 0} [a_k^-, a_k^+] \subset L^2(0, \ell)$  at  $\nu = 0.47617$ .

$k$	$a_k^-$	$a_k^+$
0	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
1	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
2	$-4.8574692133901 \times 10^{-3}$	$+4.8574692133901 \times 10^{-3}$
3	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
4	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
5	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
6	$-2.43156730049 \times 10^{-5}$	$+2.43156730050 \times 10^{-5}$
7	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
8	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
9	$-2.43151131902 \times 10^{-5}$	$+2.43151131903 \times 10^{-5}$
$k \geq 10$	$-1.0/k^4$	$1.0/k^4$

Table 4: Equilibria at  $\nu_- = 0.62163$

$k$	$u_{0+}^{\nu_-}$	$u_{1+}^{\nu_-}$	$u_2^{\nu_-}$
0	0.0	0.0	0.0
1	$-3.118023834 \times 10^{-1}$	0.0	0.0
2	0.0	$+2.202763834 \times 10^{-1}$	0.0
3	$+3.878912383 \times 10^{-3}$	0.0	0.0
4	0.0	0.0	0.0
5	$-1.2223756891 \times 10^{-5}$	0.0	0.0
6	0.0	$-5.3018430123 \times 10^{-5}$	0.0
7	$+4.5217383208 \times 10^{-8}$	0.0	0.0
8	0.0	0.0	0.0
9	$-1.4477838859 \times 10^{-10}$	0.0	0.0
$\geq 10$	0.0	0.0	0.0
$r_i^{\nu_-}$	$4.44480796087 \times 10^{-10}$	$2.097064592 \times 10^{-7}$	$7.504217330160 \times 10^{-20}$

Note that  $u_{0-}^{\nu_-} = S_0 u_{0+}^{\nu_-}$ ,  $u_{1-}^{\nu_-} = S_0 u_{1+}^{\nu_-}$ ,  $r_{0-}^{\nu_-} = r_{0+}^{\nu_-}$  and  $r_{1-}^{\nu_-} = r_{1+}^{\nu_-}$ .

Table 5: Equilibria at  $\nu_+ = 0.62173$

$k$	$u_{0+}^{\nu_+}$	$u_{1+}^{\nu_+}$
0	0.0	0.0
1	$-3.1185711867 \times 10^{-1}$	0.0
2	0.0	$+2.2035205748 \times 10^{-1}$
3	$+3.8809726153 \times 10^{-3}$	0.0
4	0.0	0.0
5	$-1.2234260453 \times 10^{-5}$	0.0
6	0.0	$-5.3073114277 \times 10^{-5}$
7	$+4.5271749737 \times 10^{-8}$	0.0
8	0.0	0.0
9	$-1.4500211168 \times 10^{-10}$	0.0
$\geq 10$	0.0	0.0
$r_i^{\nu_+}$	$4.44641647435 \times 10^{-10}$	$1.072668493 \times 10^{-7}$

$k$	$u_{1++}^{\nu_+}$	$u_{2+}^{\nu_+}$
0	$-9.6741848044 \times 10^{-6}$	0.0
1	$-2.2595797961 \times 10^{-3}$	0.0
2	$+2.2032933574 \times 10^{-1}$	0.0
3	$+4.3716007584 \times 10^{-5}$	0.0
4	$-5.5664310728 \times 10^{-8}$	0.0
5	$+3.6100856557 \times 10^{-6}$	0.0
6	$-5.3055981483 \times 10^{-5}$	0.0
7	$-1.6815825912 \times 10^{-8}$	0.0
8	$+2.6067804806 \times 10^{-11}$	0.0
9	$-6.1988247982 \times 10^{-10}$	0.0
$\geq 10$	0.0	0.0
$r_i^{\nu_+}$	$4.260517558 \times 10^{-7}$	$7.499064396468 \times 10^{-18}$

Note that  $u_{0-}^{\nu_+} = S_0 u_{0+}^{\nu_+}$ ,  $u_{1-}^{\nu_+} = S_0 u_{1+}^{\nu_+}$ ,  $u_{1+-}^{\nu_+} = S_1 u_{1++}^{\nu_+}$ ,  $u_{1-+}^{\nu_+} = S_0 u_{1++}^{\nu_+}$  and  $u_{1--}^{\nu_+} = S_1 u_{1-+}^{\nu_+}$ . Moreover,  $r_{0-}^{\nu_+} = r_{0+}^{\nu_+}$ ,  $r_{1-}^{\nu_+} = r_{1+}^{\nu_+}$  and  $r_{1+-}^{\nu_+} = r_{1++}^{\nu_+} = r_{1-+}^{\nu_+} = r_{1--}^{\nu_+}$ .

Table 6: The block  $J_2$  defining a local flow

$k$	$a_k^-$	$a_k^+$
0	$-1.818042580836000 \times 10^{-1}$	$+1.818042580836000 \times 10^{-1}$
1	$-3.368056886438130 \times 10^{-1}$	$+3.368056886438130 \times 10^{-1}$
2	$-2.423873812215343 \times 10^{-1}$	$+2.423873812215343 \times 10^{-1}$
3	$-3.10477844795332 \times 10^{-2}$	$+3.10477844795332 \times 10^{-2}$
4	$-4.8171606652801 \times 10^{-3}$	$+4.8171606652801 \times 10^{-3}$
5	$-1.1011234585183 \times 10^{-3}$	$+1.1011234585183 \times 10^{-3}$
6	$-5.348203323881 \times 10^{-4}$	$+5.348203323881 \times 10^{-4}$
7	$-4.42867581713 \times 10^{-5}$	$+4.42867581713 \times 10^{-5}$
8	$-4.26077823605 \times 10^{-5}$	$+4.26077823605 \times 10^{-5}$
9	$-1.06667909570 \times 10^{-5}$	$+1.06667909570 \times 10^{-5}$

Table 7: The Conley index of each equilibrium

Conley index	$n = 0$	$n = 1$	$n = 2$
$CH_n(M_{0+}^{\nu-})$	$\mathbb{Z}_2$	0	0
$CH_n(M_{0-}^{\nu-})$	$\mathbb{Z}_2$	0	0
$CH_n(M_{1+}^{\nu-})$	0	$\mathbb{Z}_2$	0
$CH_n(M_{1-}^{\nu-})$	0	$\mathbb{Z}_2$	0
$CH_n(M_2^{\nu-})$	0	0	$\mathbb{Z}_2$
$CH_n(M_{0+}^{\nu+})$	$\mathbb{Z}_2$	0	0
$CH_n(M_{0-}^{\nu+})$	$\mathbb{Z}_2$	0	0
$CH_n(M_{1+}^{\nu+})$	$\mathbb{Z}_2$	0	0
$CH_n(M_{1++}^{\nu+})$	0	$\mathbb{Z}_2$	0
$CH_n(M_{1+-}^{\nu+})$	0	$\mathbb{Z}_2$	0
$CH_n(M_{1+}^{\nu+})$	$\mathbb{Z}_2$	0	0
$CH_n(M_{1-+}^{\nu+})$	0	$\mathbb{Z}_2$	0
$CH_n(M_{1--}^{\nu+})$	0	$\mathbb{Z}_2$	0
$CH_n(M_2^{\nu+})$	0	0	$\mathbb{Z}_2$

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